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Approximation of parameter-dependent leadfield matrices using tensor formats

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Abstract: The source localization in the human brain depends on different uncertain parameters, e.g., the conductivity in the head. The computation of the leadfield matrix for all parameter combinations with classical methods leads to an arithmetic effort, that grows exponentially and thus renders the consideration of parameters infeasible. Using hierarchical tensor formats the linear system, which needs to be solved for the leadfield matrix, is represented in a parameter-dependent way. A linear solver within the format computes the parameter-dependent leadfield matrix and thus avoids the exponential dependency. Numerical experiments indicate a fast and accurate approximation of the leadfield matrices.

The electroencephalography

Electroencephalography (EEG) is a diagnostic procedure, e.g., for epilepsy. It involves attaching measuring sensors to the scalp of the patient to noninvasively locate the signal source in the brain that triggers the diseases. Some of the additional information required, such as the conductivity tissue in the head, is often only roughly known. Since the problem is unstable, i.e., small parameter changes lead to large deviations from the predicted source location, the accurate localization is a challenging problem. Methods to solve this inverse problem [2] often rely on the accurate solution of the corresponding forward problem.

The EEG forward problem

In the forward problem [3] the primary current source density j^p is known and one wants to calculate the electric potential u in a head domain $\Omega \subset \mathbb{R}^3$. The following partial differential equations describe this mathematically:

$$\begin{aligned} \nabla \cdot (\sigma \nabla u) &= \nabla \cdot j^p \quad && \text{in } \Omega, \\ \sigma \partial_{\vec{n}} u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\partial\Omega$ is the domain surface with normal vector $\vec{n}$ and conductivity tensor σ. We assume that the head is divided into disjoint subdomains with piecewise constant conductivity.

Since for realistic head geometries the analytical solution of (1) is unknown, we discretize the problem and approximate the solution numerically, e.g., using finite element methods [5]. This leads to a discrete representation of (1) as linear system $Ax = b$ with

$$A_{i,j} = a_h(\psi_j, \psi_i) = \int_\Omega \langle \nabla \psi_j(y), \sigma(y) \nabla \psi_i(y) \rangle \, \mathrm{d}y \tag{2}$$

and $b_i = \ell_h(\psi_i) = \int_\Omega f \psi_i \, \mathrm{d}y$, where $(\psi_i)_i$ are the test functions used in the finite element method.

The leadfield matrix

Since solving the inverse problem depends only on the solution of (1) at the measuring sensors, computing the leadfield matrix L reduces the computational effort [6]. Based on a linear map R and the transfer matrix T it holds:

$$L = Tb = RA^{-1}b = Rx = u^{\text{eeg}}.$$

As A is symmetric we can solve $AT^T = R^T$ to get the transfer matrix and with a multiplication the leadfield matrix $L = Tb$, which then can be used to solve the inverse problem for a fixed conductivity. But due to the ill-posedness of the inverse problem, we would like to consider variable conductivity values.

Parameter-dependent leadfield matrices

As Ω consists of d subdomains $(\Omega_k)_k$ with constant conductivity, for (2) it holds:

$$A_{i,j} = \sum_{k=1}^{d} \sigma_k \int_{\Omega_k} \langle \nabla \psi_j(y), \nabla \psi_i(y) \rangle \, \mathrm{d}y = \sum_{k=1}^{d} \sigma_k A^k_{i,j},$$

where A^k is independent of the conductivity. Choosing n values for all d local conductivities σ_k means that we would need to solve n^d linear systems of the form $\sum_{k=1}^{d} \sigma_k A^k T^T(\sigma) = R^T$. Even for moderate values, e.g., $d = 10$ and $n = 10$, we would need to store and solve 10 000 000 000 linear systems, which renders this method infeasible. One way to overcome this problem was presented in [4]. Based on the reduced basis

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method an approach was used to approximate the parameter-dependent leadfield matrix.

Our approach here uses low-rank tensor formats, where we can prove the following data-sparse representation of all linear systems simultaneously with

$$\mathscr{A} = \sum_{k=1}^{r} \bigotimes_{\mu=1}^{d} A_\mu^k, \qquad r = d, \tag{3}$$

$$\text{where } A_\mu^k = \begin{cases} A^k & \text{if } \mu = d, \\ \operatorname{diag}(\sigma_k^{(1)}, \dots, \sigma_k^{(n_k)}) & \text{if } \mu + k = d, \\ \mathrm{Id}_{n_{d-k}} & \text{otherwise.} \end{cases}$$

Similar results hold for the right-hand side. Such a representation is called a CANDECOMP/PARAFAC (CP) representation with representation rank r.

Low-rank tensor formats

In the CP format a tensor with representation rank r of the form (3) has a storage cost in $\mathscr{O}(dnr)$. This means that for our previous example of $d = 10$, $r = 10$ and $n = 10$ we only need to store 1000 values.

Unfortunately arithmetic operations in such formats lead to a growth of the representation rank, and therefore we need a truncation down to smaller rank. For this reason we will use the hierarchical Tucker format, in which most arithmetic operations have linear complexity in the number of parameters d, cf. [1]. Because of the error controlled truncation within this format, we can directly solve the linear system using a linear solver and avoid the exponential dependency on the number of parameters.

Numerical experiments

We present numerical experiments for a spherical head model, where we use a linear solver within the low-rank tensor format. After each arithmetic operation we truncate the representation of our solution with a relative accuracy of $\varepsilon = 1 \times 10^{-9}$. As a first experiment we compare the runtime of our method for $d = 4$ parameters and different number of values n against the classical method in Fig. 1. We observe that the time to compute all solutions in the tensor format depends only weakly on n, while we observe an exponential growth in the runtime, if we would use classical methods. As a second experiment we compare the reference solution u^{ref}, calculated numerically with duneuro[1] for random parameter values, with our solution for $d = 4$ and different n in Tab. 1. There we use the relative difference measure (RDM) given by $\text{RDM}(u^{\text{num}}, u^{\text{ref}}) = \|\frac{u^{\text{num}}}{\|u^{\text{num}}\|_2} - \frac{u^{\text{ref}}}{\|u^{\text{ref}}\|_2}\|_2$ and the logarithmic magnitude error (lnMAG) given by $\text{lnMAG}(u^{\text{num}}, u^{\text{ref}}) = \log(\frac{\|u^{\text{num}}\|_2}{\|u^{\text{ref}}\|_2})$ as error measures. We observe that for varying values of n the error measure is small, which indicates that our solution is accurate.

1 http://www.duneuro.org

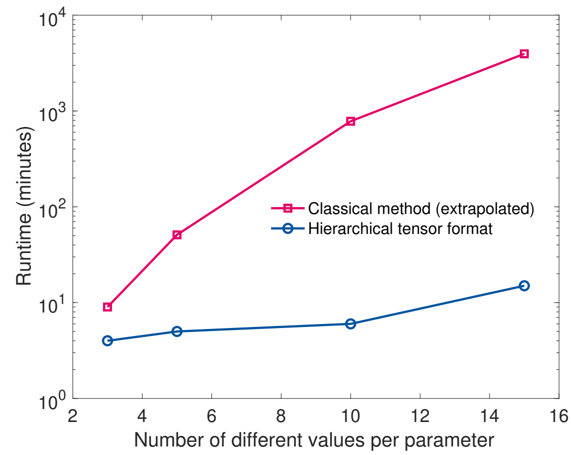

Fig. 1: Comparison of the runtime to compute all leadfield matrices for 4 parameter

Tab. 1: Sampled mean error measures of the approximated tensor format solution with 4 parameter

Values per parameter	RDM	lnMAG
3	6.04×10^{-8}	2.79×10^{-8}
5	8.43×10^{-7}	-5.03×10^{-8}
10	5.67×10^{-5}	1.01×10^{-5}
15	1.05×10^{-4}	-1.56×10^{-5}

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