Mathematical Methods for the Registration of Medical Images: Part I (parametric approaches)

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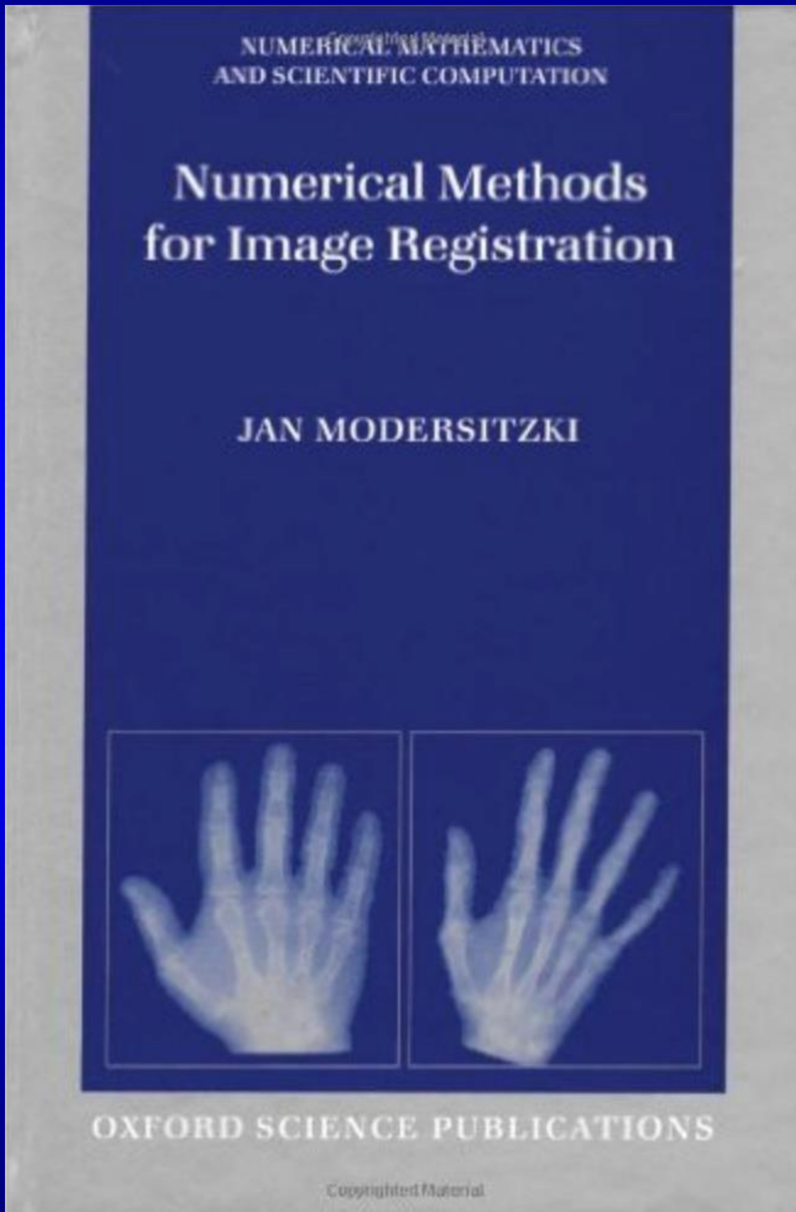
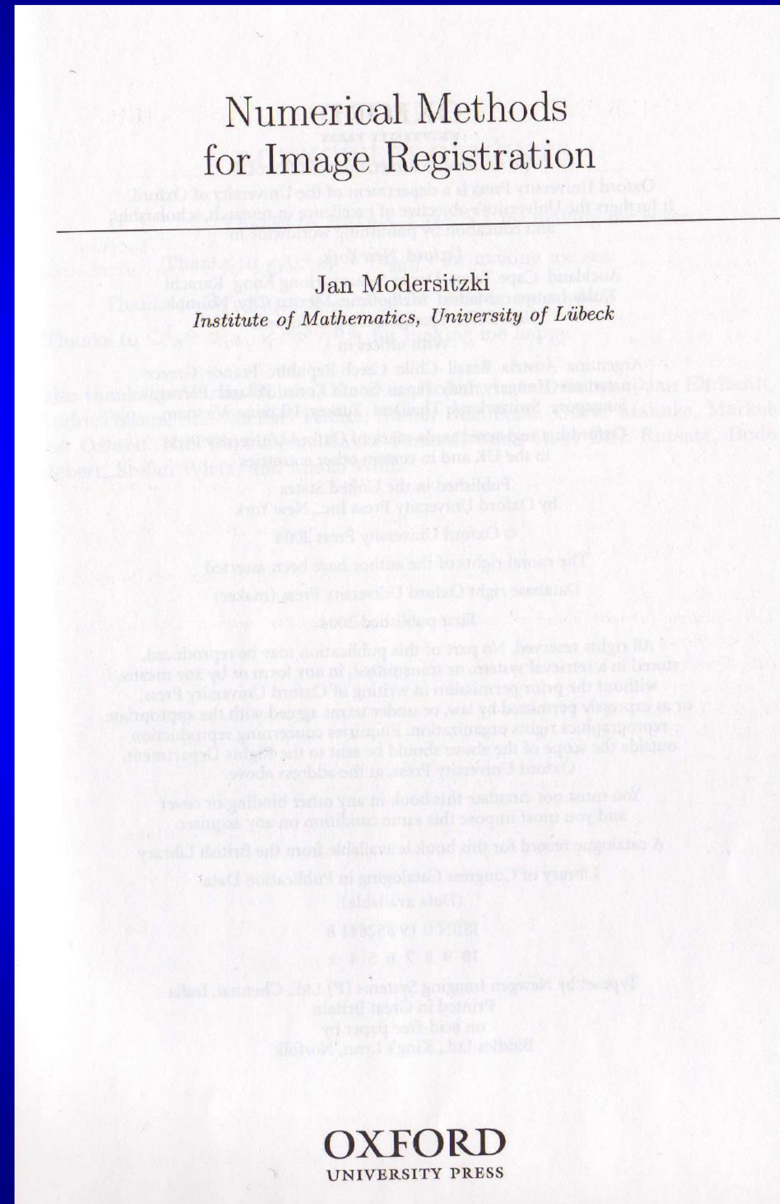
[Modersitzki, Numerical Methods for Image Registration, *Oxford University Press, 2004*]

Literature





Basic definitions

Definition 1.1. *Let $d \in \mathbb{N}$. A function $b : \mathbb{R}^d \to \mathbb{R}$ is called a d-dimensional image, if*

1. *b is compactly supported,*
2. $0 \leq b(x) < \infty \quad \forall x \in \mathbb{R}^d$,
3. *$\int_{\mathbb{R}^d} b^k(x)dx$ is finite for $k > 0$.*

The set of all images is denoted by

$$\mathrm{Img}(d) := \{b : \mathbb{R}^d \to \mathbb{R} | b \text{ is } d - dimensional \text{ image}\}.$$

Basic definitions

Definition 1.2. *Let $d \in \mathbb{N}$, $\Omega :=]0,1[^d$, and $n_1, \ldots, n_d \in \mathbb{N}$ be some given numbers. The points*

$$x_{j_1,\ldots,j_d} = (x_{j_1}, \ldots, x_{j_d})^T \in \Omega \cup \partial\Omega,$$

where $1 \le j_l \le n_l$ and $1 \le l \le d$, are called grid points. *The array*

$$X := (x_{j_1,\ldots,j_d})_{\substack{1 \le j_l \le n_l \\ l=1,\ldots,d}} \in \mathbb{R}^{n_1 \times \ldots \times n_d}$$

is called the grid matrix.

Let $N := \prod_{i=1}^{d} n_i$ and let the numbers $j \in \mathbb{N}(1 \le j \le N)$ and $(j_1, \ldots, j_d) \in \mathbb{N}^d$ $(1 \le j_l \le n_l,\ 1 \le l \le d)$ be related by a one-to-one lexicographical ordering, $j = \sum_{\nu=1}^{d-1} (j_{\nu+1} - 1) \prod_{\mu=1}^{\nu} n_\mu + j_1$. The vector $\vec{X} := (x_j)_{j=1,\ldots,N} \in \mathbb{R}^N$, where $x_j = x_{j_1,\ldots,j_d}$, is called the grid vector. *The set $\Omega_d := \{x_j, j = 1, \ldots, N\}$ is called an $n_1 \times \ldots \times n_d$ grid.*

Basic definitions

Definition 1.3. *Let* $d, q \in \mathbb{N}$ *and* $\kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{N}^d$. *The* polynomials *of degree* q *are defined by*

$$\Pi_q(\mathbb{R}^d) := \left\{ \psi : \mathbb{R}^d \to \mathbb{R} \mid \psi(x) = \sum_{|\kappa| \leq q} \alpha_\kappa x^\kappa,\ \alpha_\kappa \in \mathbb{R} \right\}$$

where $|\kappa| = \kappa_1 + \ldots + \kappa_d$ *and for* $x \in \mathbb{R}^d$ *we set* $x^\kappa := x_1^{\kappa_1} \cdot \ldots \cdot x_d^{\kappa_d}$. *The set of* d*-dimensional polynomials of degree* q *is defined by*

$$\Pi_q^d(\mathbb{R}^d) := \left\{ \varphi : \mathbb{R}^d \to \mathbb{R}^d \mid \varphi_l \in \Pi_q(\mathbb{R}^d),\ l = 1, \ldots d \right\}.$$

Structure

- Parametric image registration techniques
- Non-parametric image registration techniques
- Non-parametric registration for DTI

Structure

- Parametric image registration techniques
  - Landmark-based parametric registration
  - Landmark-based smooth registration
  - Parametric registration techniques
    - Intensity-based registration
    - Mutual information-based registration

Landmark-based parametric registration: The problem

$$\mathcal{D}^{LM}[\varphi] = \sum_{j=1}^{m} \|x^{R,j} - \varphi(x^{T,j})\|_2^2$$

$$\varphi_l = \sum_{k=1}^{n} \alpha_{l,k}\psi_k, \qquad \alpha_{l,k} \in \mathbb{R}, \quad \psi_k : \mathbb{R}^d \to \mathbb{R}, \quad n \in \mathbb{N}, \quad l = 1, \ldots, d$$

Reference image

Template image

Landmark-based parametric registration: The problem

$$\mathcal{D}^{LM}[\varphi] = \sum_{j=1}^{m} \|x^{R,j} - \varphi(x^{T,j})\|_2^2$$

$$\varphi_l = \sum_{k=1}^{n} \alpha_{l,k}\psi_k, \qquad \alpha_{l,k} \in \mathbb{R}, \quad \psi_k : \mathbb{R}^d \to \mathbb{R}, \quad n \in \mathbb{N}, \quad l = 1,\dots,d$$

Problem 1.4. *For* $\varphi = (\varphi_1,\dots,\varphi_d)$, *find parameters* $\alpha_{l,k} \in \mathbb{R}$, $k = 1,\dots,n$, $l = 1,\dots,d$, *such that* $\mathcal{D}^{LM}[\varphi] \to \min!$.

Landmark-based parametric registration: The solution

$$\mathcal{D}^{\mathrm{LM}}[\varphi] = \sum_{j=1}^{m} \sum_{\ell=1}^{d} \left(x_\ell^{R,j} - \sum_{k=1}^{n} \alpha_{\ell,k} \psi_k(x^{T,j}) \right)^2 = \sum_{\ell=1}^{d} \| y_\ell - \Psi a_\ell \|_{\mathbb{R}^d}^2$$

$$y_\ell = (x_\ell^{R,1}, \ldots, x_\ell^{R,m})^\top \in \mathbb{R}^m,$$

$$\Psi = \left(\psi_k(x^{T,j}) \right)_{\substack{j=1,\ldots,m \\ k=1,\ldots,n}} \in \mathbb{R}^{m \times n},$$

$$a_\ell = (\alpha_{\ell,1}, \ldots, \alpha_{\ell,n})^\top \in \mathbb{R}^n.$$

$$\varphi = \begin{pmatrix} \sum_{k=1}^{d+1} \alpha_{1,k} \psi_k \\ \vdots \\ \sum_{k=1}^{d+1} \alpha_{d,k} \psi_k \end{pmatrix} \in \Pi_1^d(\mathbb{R}^d)$$

$$\psi_1(x) = 1,\ \psi_{\ell+1}(x) = x_\ell,\ \ell = 1, \ldots, d.$$

Landmark-based parametric registration when using linear polynomial

Reference image

Template image

Result of landmark registration (Fig.4.1, linear)

Landmark-based parametric registration when using quadratic polynomial

Reference image

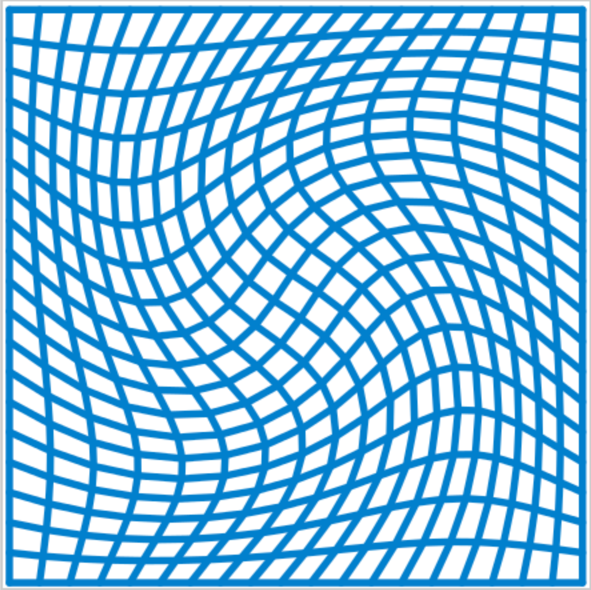
Template image

Result of landmark Registration (Fig.4.1, quadratic)

Diffeomorphism (from Wikipedia)

Definition [edit]

Given two manifolds M and N, a differentiable map $f : M \to N$ is called a **diffeomorphism** if it is a bijection and its inverse $f^{-1} : N \to M$ is differentiable as well. If these functions are r times continuously differentiable, f is called a **C^r-diffeomorphism**.



The image of a rectangular grid on a square under a diffeomorphism from the square onto itself.

Landmark-based parametric registration

interpolation.

The main disadvantage is, however, that the transformation from the parametric approach is in general not diffeomorphic. Figure 4.1 shows the results for a linear ($\varphi \in \Pi_1^d(\mathbb{R}^d)$) and a quadratic ($\varphi \in \Pi_2^d(\mathbb{R}^d)$) parametric registration. As is apparent from this figure, the linear approach yields satisfactory results, though the fit of the landmarks is not perfect. After quadratic registration, all landmarks are mapped perfectly. However, since φ is a quadratic polynomial, the map is not diffeomorphic and leads to a "mirrored" image, which is certainly not a satisfactory registration. Note that in this example, the landmarks are chosen such that the interpolation problem is well-posed.

Structure

- Parametric image registration techniques
  - Landmark-based parametric registration
  - Landmark-based smooth registration (only of academic interest for us)
  - Parametric registration techniques
    - Intensity-based registration
    - Mutual information-based registration

Landmark-based smooth registration: Basic definitions

$$\langle f, g \rangle_0 := \langle f, g \rangle_{L_2} := \int_{\mathbb{R}^d} f(x) g(x) dx$$

$$\langle \cdot\,, \cdot \rangle_q : \mathcal{X} \times \mathcal{X} \to \mathbb{R}, \quad \langle f, g \rangle_q := \sum_{|\kappa| = q} c_\kappa \langle D^\kappa f, D^\kappa g \rangle_0$$

$$D^\kappa f = \left(\frac{\partial}{\partial x_1}\right)^{\kappa_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\kappa_d} f$$

$$\mathcal{X} = H^q \cap C(\mathbb{R}^d) \cup \Pi_{q-1}(\mathbb{R}^d)$$

Landmark-based smooth registration: Basic definitions

The set of coefficients $\{ c_\kappa : |\kappa| = q \}$ is chosen such that the semi-norm is rotationally invariant. Explicitly, these parameters are specified via the formal expansion

$$\|x\|_{\mathbb{R}^d}^{2q} = \sum_{|\kappa|=q} c_\kappa x^{2\kappa} = \sum_{|\kappa|=q} c_\kappa x_1^{2\kappa_1} \cdots x_d^{2\kappa_d} \,. \tag{4.7}$$

In particular for $d = q = 2$, we have

$$\|x\|_{\mathbb{R}^2}^4 = \left(x_1^2 + x_2^2\right)^2 = x_1^4 + 2x_1^2 x_2^2 + x_2^4$$

and

$$\langle f, g\rangle_2 = \int_{\mathbb{R}^d} \partial_{x_1x_1} f \, \partial_{x_1x_1} g + 2\partial_{x_1x_2} f \, \partial_{x_1x_2} g + \partial_{x_2x_2} f \, \partial_{x_2x_2} g \, dx.$$

Landmark-based smooth registration: Basic definitions

interpolation data $(x_j, y_j) \in \mathbb{R}^d \times \mathbb{R},\ j = 1, \ldots, m$

$$[p \in \Pi_{q-1}(\mathbb{R}^d) \wedge p(x_j) = 0,\ \ j = 1, \ldots, m] \Rightarrow p \equiv 0$$

$$m \geq d_q := \dim(\Pi_{q-1}(\mathbb{R}^d))$$

$$[f, g]_q := \langle f, g \rangle_q + \sum_{j=1}^{m} f(x_j) g(x_j)$$

$$H := (\mathcal{X}, [\cdot\,, \cdot]_q)$$

[Rohr, *Computational Imaging and Vision, 2001*]
[Modersitzki, Numerical Methods for Image Registration, *Oxford University Press, 2004*]

Landmark-based smooth registration: Basic theory

Theorem 4.2 *The radial basis function for $(-1)^q \sum_{|\kappa|=q} c_\kappa D^{2\kappa}$ is given by*

$$\rho(r) := c_q^d \begin{cases} r^{2q-d} \log r, & d \text{ even}, \\ r^{2q-d}, & d \text{ odd}, \end{cases} \tag{4.14}$$

where

$$c_q^d = \begin{cases} \frac{(-1)^{q+1+d/2}}{2^{2q-1}\pi^{d/2}(q-1)!(q-d/2)!}, & d \text{ even}, \\ \frac{\Gamma(d/2-q)}{2^{2q}\pi^{d/2}(q-1)!}, & d \text{ odd}. \end{cases}$$

$$\rho_x(y) := \rho(\|y-x\|_{\mathbb{R}^d})$$

$$(-1)^q \sum_{|\kappa|=q} c_\kappa D^{2\kappa} \rho_x = \delta_x$$

[Light, *Numerical Analysis, 1995*]
[Modersitzki, Numerical Methods for Image Registration, *Oxford University Press, 2004*]

Landmark-based smooth registration: Basic theory

Theorem 4.5 *Let $d, m, q \in \mathbb{N}$ and $d_q := \dim(\Pi_{q-1}(\mathbb{R}^d))$ and let $x_j \in \mathbb{R}^d$, $y_j \in \mathbb{R}$, $j = 1, \ldots, m$, be given interpolation data. The minimal norm solution*

$$\psi = \arg\min\{[f, f]_q,\ f \in H \text{ and } f(x_j) = y_j,\ j = 1, \ldots, m\}$$

is characterized by

$$\psi = \sum_{j=1}^{m} \theta_j \rho_{x_j} + \sum_{j=1}^{d_q} \beta_j p_j, \tag{4.16}$$

where $\rho_{x_j} = \rho(\|\cdot - x_j\|_{\mathbb{R}^d})$ (see Theorem 4.2) and $p_1, \ldots, p_{d_q}$ is a basis for $\Pi_{q-1}(\mathbb{R}^d)$. The coefficients $\theta := (\theta_1, \ldots, \theta_m)^\top \in \mathbb{R}^m$ and $\beta := (\beta_1, \ldots, \beta_{d_q})^\top \in \mathbb{R}^{d_q}$ are determined by the following system of linear equations,

$$\begin{pmatrix} K & B^\top \\ B & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \beta \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}, \tag{4.17}$$

$$y := (y_1, \ldots, y_m)^\top \in \mathbb{R}^m,$$

$$K := (\rho(\|x_j - x_k\|_{\mathbb{R}^d}))_{j,k=1,\ldots,m} \in \mathbb{R}^{m \times m}, \tag{4.18}$$

$$\text{and} \quad B := (p_j(x_k))_{\substack{j=1,\ldots,d_q \\ k=1,\ldots,m}} \in \mathbb{R}^{d_q \times m}. \tag{4.19}$$

Landmark-based smooth registration: The problem

Thin Plate Spline (TPS) interpolation

$$\mathcal{D}^{\mathrm{LM}}[\varphi] = \sum_{j=1}^{m} \left\| x^{R,j} - \varphi(x^{T,j}) \right\|_{\mathbb{R}^d}^2$$

$$\mathcal{S}^{\mathrm{TPS}}[\varphi] := \frac{1}{2} \sum_{\ell=1}^{d} \langle \varphi_\ell, \varphi_\ell \rangle_q$$

Moreover, we define the interpolation space

$$\mathrm{IS}_{q,m} := \operatorname{span}\{\rho(\|x - x^{T,j}\|_{\mathbb{R}^d}),\ j = 1, \ldots, m\}^d \cup \Pi_{q-1}^d(\mathbb{R}^d). \qquad (4.27)$$

Problem 4.6 *Given $\alpha > 0$, find a transformation $\varphi : \mathbb{R}^d \to \mathbb{R}^d$, $\varphi \in \mathrm{IS}_{q,m}$, such that*

$$\alpha \mathcal{S}^{\mathrm{TPS}}[\varphi] + \mathcal{D}^{\mathrm{LM}}[\varphi] \longrightarrow \min,$$

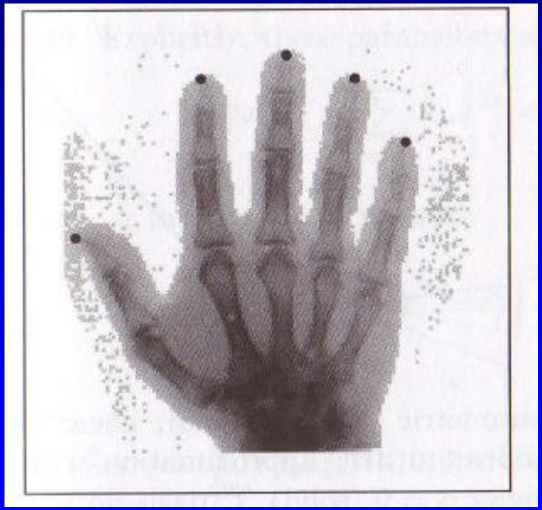
Landmark-based smooth registration: The solution

$$\varphi_\ell = \sum_{j=1}^{m} \theta_{\ell,j} \rho_{x_j} + \sum_{j=1}^{d_q} \beta_{\ell,j} p_j, \quad \ell = 1, \dots, d$$

$$\begin{pmatrix} K + \alpha I & B^\top \\ B & 0 \end{pmatrix} \begin{pmatrix} \theta_\ell \\ \beta_\ell \end{pmatrix} = \begin{pmatrix} y_\ell \\ 0 \end{pmatrix}$$

$$y_\ell := (x_\ell^{R,1}, \dots, x_\ell^{R,m})^\top \in \mathbb{R}^m$$

Landmark-based smooth registration: An example

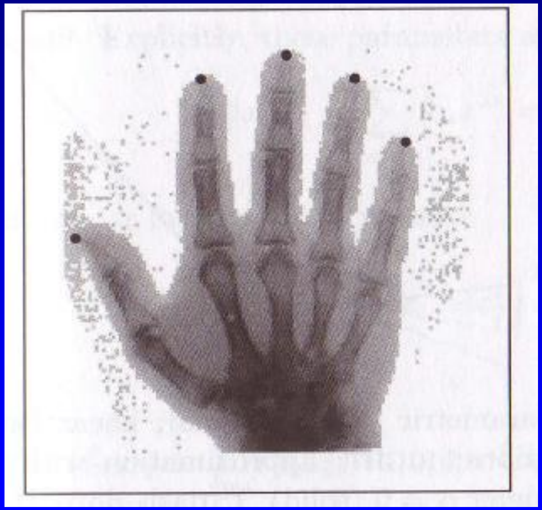


Reference image

Template image

Result for alpha=10^5 (Fig.4.4)

Landmark-based smooth registration: An example



Reference image

Template image

Result for alpha=0 (Fig.4.4)

Landmark-based smooth registration

Figure 4.4 displays results for the landmark-based registration of reference and template images which have already been shown in Fig. 4.1. As is apparent from this figure, the registration ranges from an interpolation of the landmarks ($\alpha = 0$) to almost affine linear registration for large values of α. Note that the registration is governed completely by the landmarks.

Landmark-based smooth registration

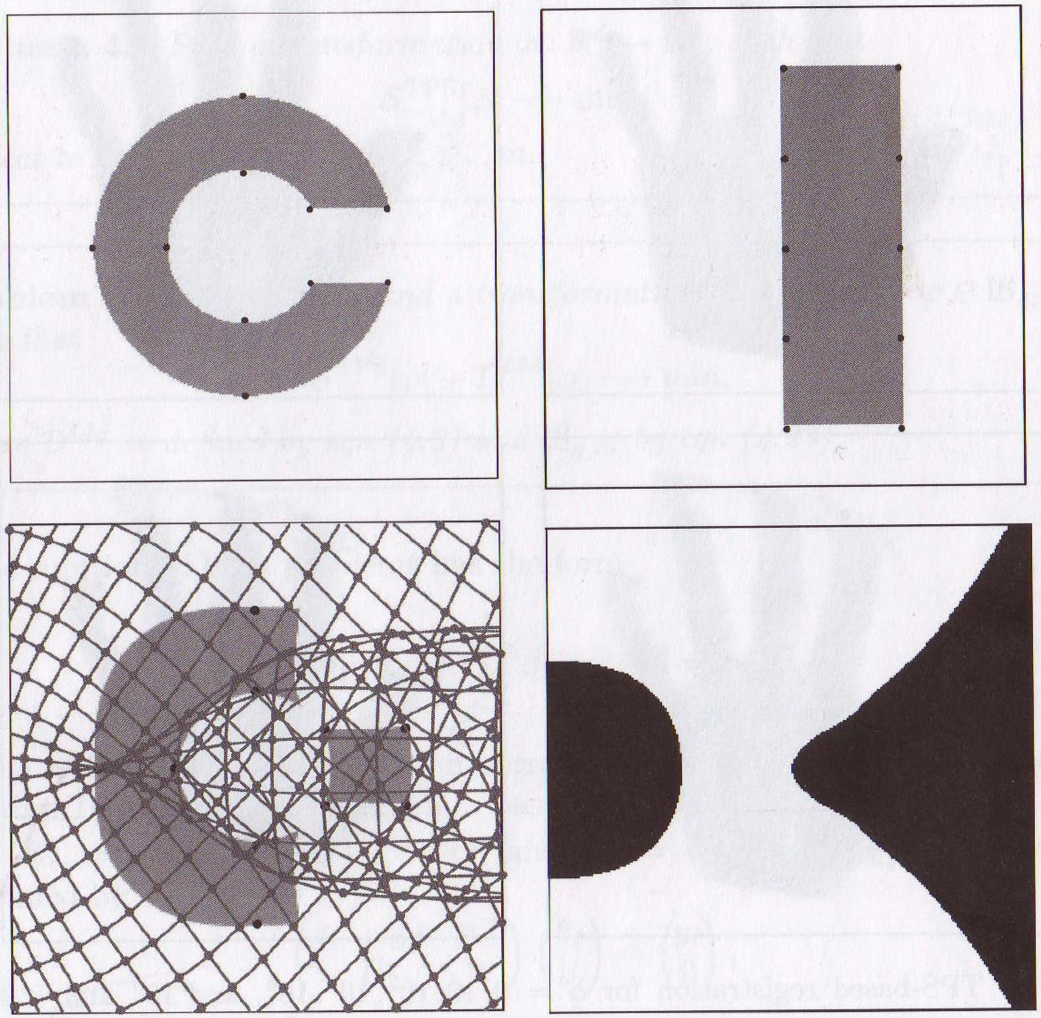


FIG. 4.5 TOP LEFT: reference image with landmarks (dots), TOP RIGHT: template image with landmarks (dots), BOTTOM LEFT: template image after TPS registration with landmarks and deformed grid, BOTTOM RIGHT: sign of determinant of Jacobian of φ (white: $\det \nabla\varphi \geq 0$, black: $\det \nabla\varphi < 0$).

Landmark-based smooth registration

Finally, Fig. 4.5 illustrates that landmark-based registration does not always result in a meaningful registration. The landmarks are chosen such that one expects a bending of the rectangular bar displayed in the template image. Note that the landmarks are chosen in a meaningful ordering. However, although the transformation is smooth, it fails to be diffeomorphic. This can be seen in the bottom right picture, where the sign of the Jacobian of the transformation, i.e., $\text{sign}(\det \nabla\varphi)$, is shown.

Structure

- Parametric image registration techniques
  - Landmark-based parametric registration
  - Landmark-based smooth registration
  - Parametric registration techniques
    - Intensity-based distance measure
    - Mutual information-based distance measure

Intensity-based registration

Definition 6.1 *Let* $d \in \mathbb{N}$ *and* $R, T \in \mathrm{Img}(d)$. *The* sum of squared differences (SSD) *distance measure* $\mathcal{D}^{\mathrm{SSD}}$ *is defined by* $\mathcal{D}^{\mathrm{SSD}} : \mathrm{Img}(d)^2 \to \mathbb{R}$,

$$\mathcal{D}^{\mathrm{SSD}}[R,T] := \frac{1}{2}\|T - R\|_{L_2}^2 = \frac{1}{2}\int_{\mathbb{R}^d} \big(T(x) - R(x)\big)^2 dx.$$

For a transformation $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ *we also define*

$$\mathcal{D}^{\mathrm{SSD}}[R,T;\varphi] = \mathcal{D}^{\mathrm{SSD}}[R, T \circ \varphi], \tag{6.2}$$

and for a parametric transformation φ_a *we set*

$$D^{\mathrm{SSD}}(R,T;a) = \mathcal{D}^{\mathrm{SSD}}[R, T \circ \varphi_a]. \tag{6.3}$$

Restricted transformations

In many applications there exist implicit requirements with respect to the transformation. A typical example is that the transformation has to be smooth or even diffeomorphic. Approaches based on regularization are investigated in Part II. Explicit parametric requirements are even more popular in the literature and typical example are given below.

Rigid transformation, $\varphi(x) = Qx + b$, where $Q \in \mathbb{R}^{d\times d}$ is orthogonal with $\det Q = 1$ and $b \in \mathbb{R}^d$. The transformation is called rigid because only rotations and translations of the coordinates are permitted.

(Affine) linear transformation, $\varphi(x) = Ax + b$, $A \in \mathbb{R}^{d\times d}$ with $\det A > 0$, $b \in \mathbb{R}^d$. Note that in contrast to other authors we allow for individual scalings.

Polynomial transformation, $\varphi \in \Pi_q^d(\mathbb{R}^d)$; cf., Definition 3.6.

B-spline transformation,

$$\varphi = (\varphi_1, \ldots, \varphi_d)^\top : \mathbb{R}^d \to \mathbb{R}^d, \quad \text{where} \quad \varphi_\ell \in \mathrm{Spline}_{y,q}(\mathbb{R}^d),$$

where $\mathrm{Spline}_{y,q}(\mathbb{R}^d)$ is a d-dimensional spline space spanned by B-splines of degree q with respect to the knots y_k, $k = 1, \ldots, K$; cf., e.g., De Boor (1978).

Intensity-based affine registration

We start by introducing a set of feasible transformations, which here are supposed to be affine linear maps, i.e., $\varphi \in \Pi_1^d(\mathbb{R}^d)$; cf., Definition 3.6. A mathematical formulation of the registration problem then reads as follows.

Problem 6.1 *Find* $\varphi \in \Pi_1^d(\mathbb{R}^d)$ *such that* $\mathcal{D}[\varphi] = \min$.

The essential point here is that the set $\in \Pi_1^d(\mathbb{R}^d)$ can be parameterized. For a specific element φ of $\in \Pi_1^d(\mathbb{R}^d)$, we make use of the notation φ_a, where

$$\varphi_{a;\ell}(x) = a_{\ell,0} + \sum_{j=1}^{d} a_{\ell,j} x_j, \quad \ell = 1, \ldots, d.$$

The parameters $a_{\ell,j}$ are gathered together in a vector,

$$a = (a_{1,0}, \ldots, a_{1,d}, \ldots, \quad a_{d,0}, \ldots, a_{d,d})^\top \in \mathbb{R}^n, \quad n = d(d+1).$$

Intensity-based affine registration

Moreover, we set

$$D(a) := \mathcal{D}[\varphi_a] \quad \text{and} \quad T_a := T \circ \varphi_a. \tag{6.1}$$

Thus, Problem 6.1 may be reformulated in terms of a parameterized finite-dimensional optimization problem.

Problem 6.2 *Find $a \in \mathbb{R}^n$, such that $D(a) = \min$.*

Intensity-based affine registration: An example

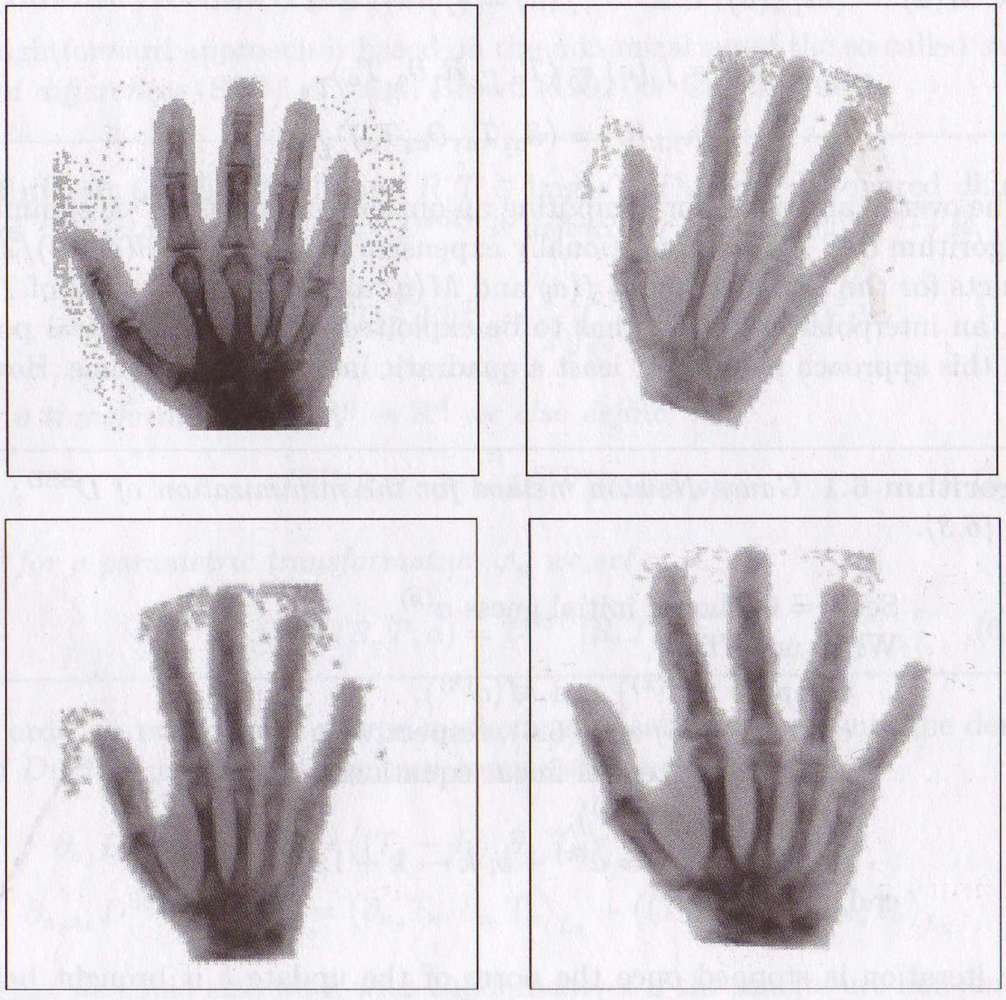


FIG. 6.1 Optimal intensity-based affine linear registration; reference (TOP LEFT), template (TOP RIGHT), template after rigid registration (BOTTOM LEFT), template after affine linear registration (BOTTOM RIGHT).

Structure

- Parametric image registration techniques
  - Landmark-based parametric registration
  - Landmark-based smooth registration
  - Parametric registration techniques
    - Intensity-based distance measure
    - Mutual information-based distance measure

Mutual information-based registration

Definition 6.6 *Let $q \in \mathbb{N}$ and ρ be a density on $\mathbb{R}^q$, i.e., $\rho : \mathbb{R}^q \to \mathbb{R}$, $\rho(x) \geq 0$, and $\int_{\mathbb{R}^q} \rho(x)\, dx = 1$. The (differential)* entropy *of the density is defined by*

$$H(\rho) := -\mathbb{E}_\rho [\log \rho] = -\int_{\mathbb{R}^q} \rho \log \rho \; dg.$$

[Collignon et al., *Computational Imaging and Vision 3, 1995*]
[Maess et al., *IEEE Trans.Med.Imag., 16 (2), 1997*]
[Modersitzki, Numerical Methods for Image Registration, *Oxford University Press, 2004*]

Mutual information-based registration

Definition 6.7 *Let* $d \in \mathbb{N}, R, T \in \mathrm{Img}(d)$. *The* mutual information (MI) *distance measure* $\mathcal{D}^{\mathrm{MI}}$ *is defined by* $\mathcal{D}^{\mathrm{MI}} : \mathrm{Img}(d)^2 \to \mathbb{R}$,

$$\mathcal{D}^{\mathrm{MI}}[R,T] := H(\rho_R) + H(\rho_T) - H(\rho_{R,T}),$$

where ρ_R, ρ_T, *and* $\rho_{R,T}$ *denote the gray-value densities of* R, T, *and the joint gray-value distribution, respectively.*

For a transformation $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ *we also define*

$$\mathcal{D}^{\mathrm{MI}}[R,T;\varphi] = \mathcal{D}^{\mathrm{MI}}[R, T \circ \varphi], \tag{6.9}$$

and for a parametric transformation φ_a *we set*

$$D^{\mathrm{MI}}(R,T;a) = \mathcal{D}^{\mathrm{MI}}[R, T \circ \varphi_a]. \tag{6.10}$$

$$\mathcal{D}^{\mathrm{MI}}[R,T] = \int_{\mathbb{R}^2} \rho_{R,T}(g_R, g_T) log \frac{\rho_{R,T}(g_R, g_T)}{\rho_R(g_R)\rho_T(g_T)} d(g_R, g_T)$$

Kullback-Leibler measure

Kullback-Leibler measure is identical to MI distance measure

$$\int_{\mathbb{R}^2} \rho_{R,T}(g_R, g_T) log \frac{\rho_{R,T}(g_R, g_T)}{\rho_R(g_R)\rho_T(g_T)} d(g_R, g_T)$$

$$= \int_{\mathbb{R}^2} (\log \rho_{R,T}(g_1, g_2) - \log \rho_R(g_1) - \log \rho_T(g_2))\rho_{R,T}(g_1, g_2)\, d(g_1, g_2)$$

$$= -H(\rho_{R,T}) - \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_{R,T}(g_1, g_2)\, dg_2 \;\log \rho_R(g_1)\, dg_1$$

$$- \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_{R,T}(g_1, g_2)\, dg_1 \;\log \rho_T(g_2)\, dg_2$$

$$= -H(\rho_{R,T}) - \int_{\mathbb{R}} \rho_R(g_1) \log \rho_R(g_1)\, dg_1 - \int_{\mathbb{R}} \rho_T(g_2) \log \rho_T(g_2)\, dg_2,$$

$$= H(\rho_R) + H(\rho_T) - H(\rho_{R,T}),$$

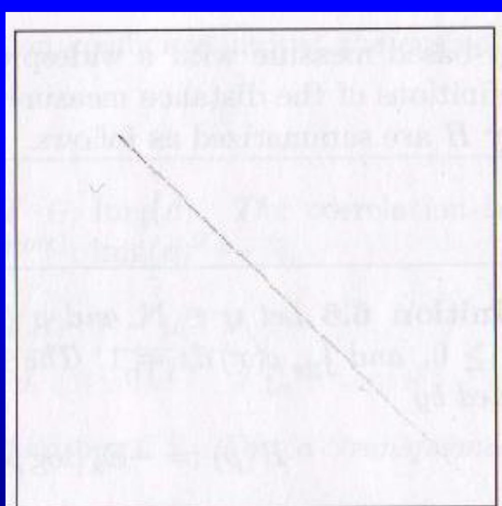
$$= \mathcal{D}^{\mathrm{MI}}[R, T]$$

Maximization of MI distance measure: An example

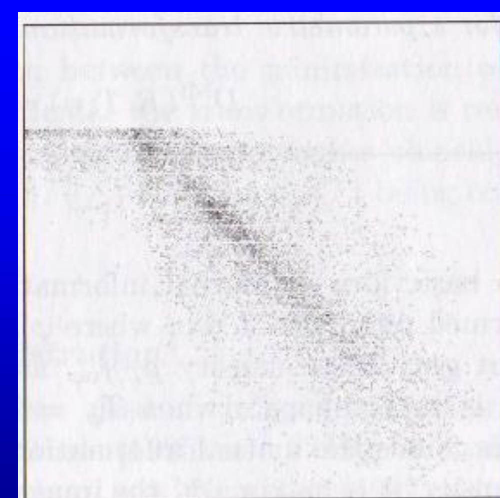
$\alpha = 0$, MI ≈ 1.709

$$\mathrm{MI} := \mathcal{D}^{\mathrm{MI}}[\mathrm{T}, \mathrm{T} \circ \varphi]$$

$\alpha = 5$, MI ≈ 0.798



$$\rho_{T,T_\varphi}$$

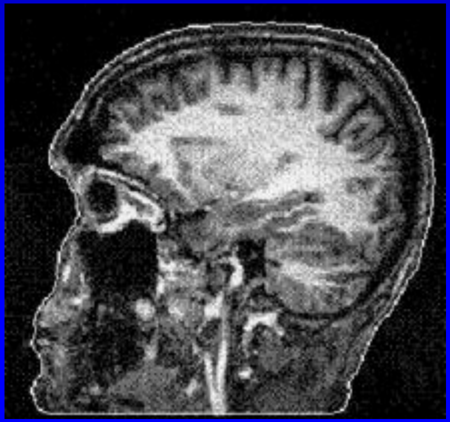


Rotation of reference by alpha=0 degree

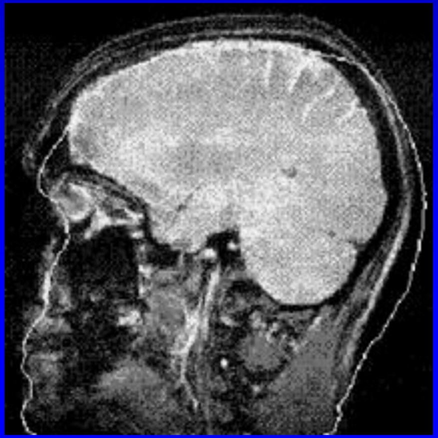
Rotation of reference by alpha=5 degree

MI-based registration for brain research: An example

T1-MRI



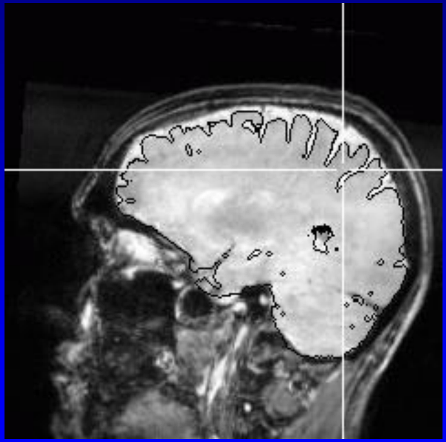
PD-MRI



T1-MRI:
Appropriate for segmentation of
- **Scalp surface**
- **Outer skull surface**
- **Gray Matter surface**
- **White Matter surface**
- **Ventricle surfaces**

Proton-Density-weighted MRI:
Appropriate for segmentation of
- **Inner skull surface.**

MI-based registration for brain research: An example



- **Parametric transformation (translation, rotation, scaling)**

  [de Munck et al., *IEEE Trans.Biomed.Eng., 15 (5), 1996*]

$$PD^{\alpha} := \varphi_{\alpha}(PD)$$

- **Distance measure: Mutual information**

  [Maess et al., *IEEE Trans.Med.Imag., 16 (2), 1997*]

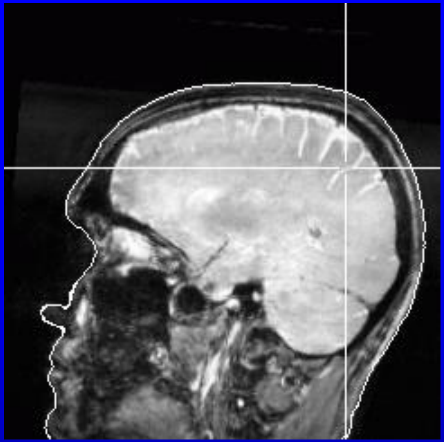
$$P(X_{\mathrm{MRI}} = g_{\mathrm{MRI}}) := \frac{h_{\mathrm{MRI}}(g_{\mathrm{MRI}})}{N}$$

$$\mathcal{D}^{\mathrm{MI}}[T1, PD, \varphi_{\alpha}] := \sum_{g_{T1};g_{PD^{\alpha}}} P(X_{T1} = g_{T1}, X_{PD^{\alpha}} = g_{PD^{\alpha}})\, log \frac{P(X_{T1} = g_{T1}, X_{PD^{\alpha}} = g_{PD^{\alpha}})}{P(X_{T1} = g_{T1})P(X_{PD^{\alpha}} = g_{PD^{\alpha}})}$$

- **Optimization: Simplex on Gauss-Pyramid**

  [Allgower and Georg, 1990]

$$\alpha^{*} = max_{\alpha}\mathcal{D}^{\mathrm{MI}}[T1, PD, \varphi_{\alpha}]$$



$$PD^{\alpha^{*}}$$

[Wolters, *lecture scriptum, chapter 4.5*]

Visual validation of MI-based registration of PD-MRI on T1-MRI

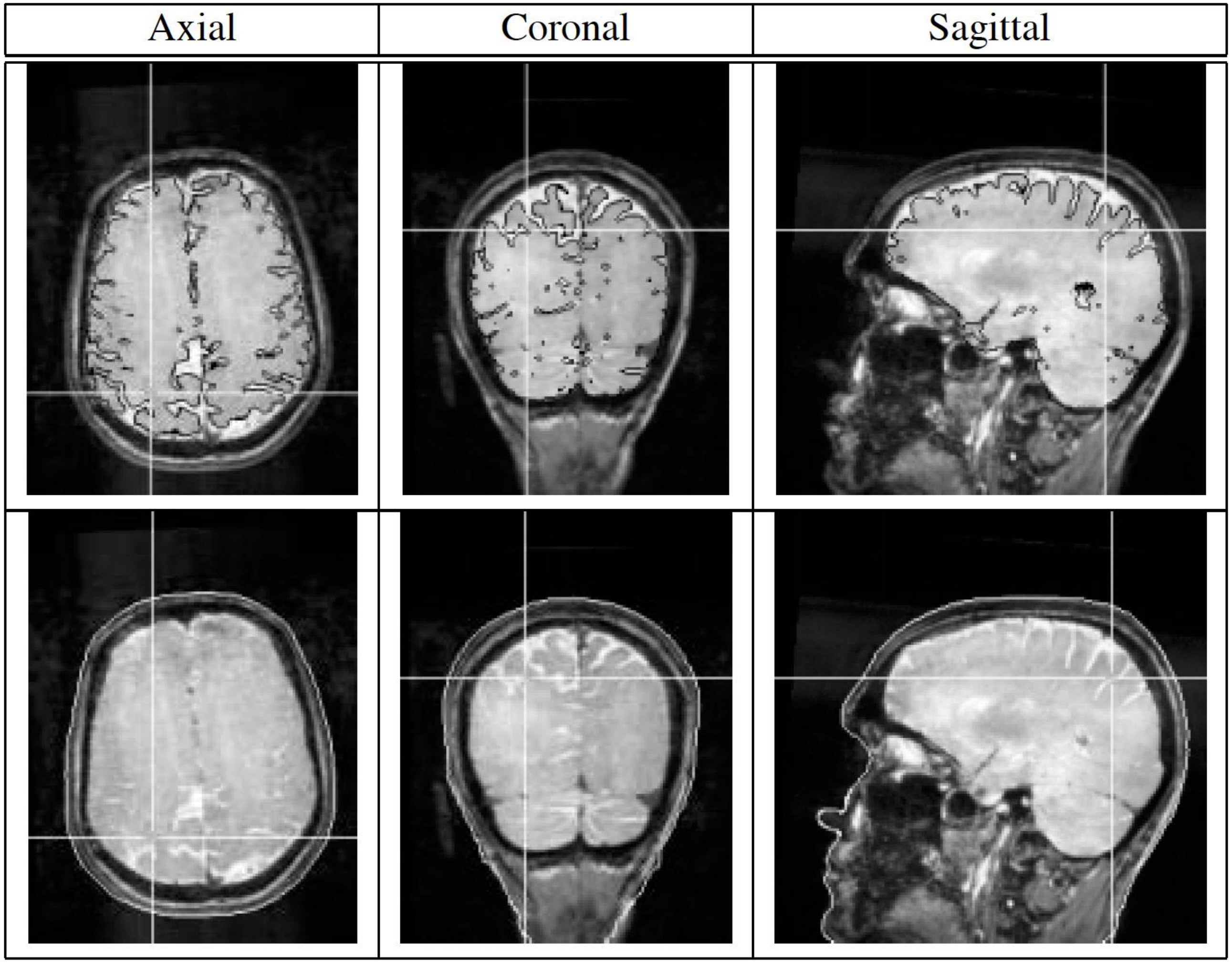


Figure 2.9: *Registration result: The outer surfaces of the brain (top) and of the head (bottom), extracted from the T1-, are mapped on the registered PD-MRI.*

Influence of MI-based registration of T1/PD-MRI on inner skull segmentation

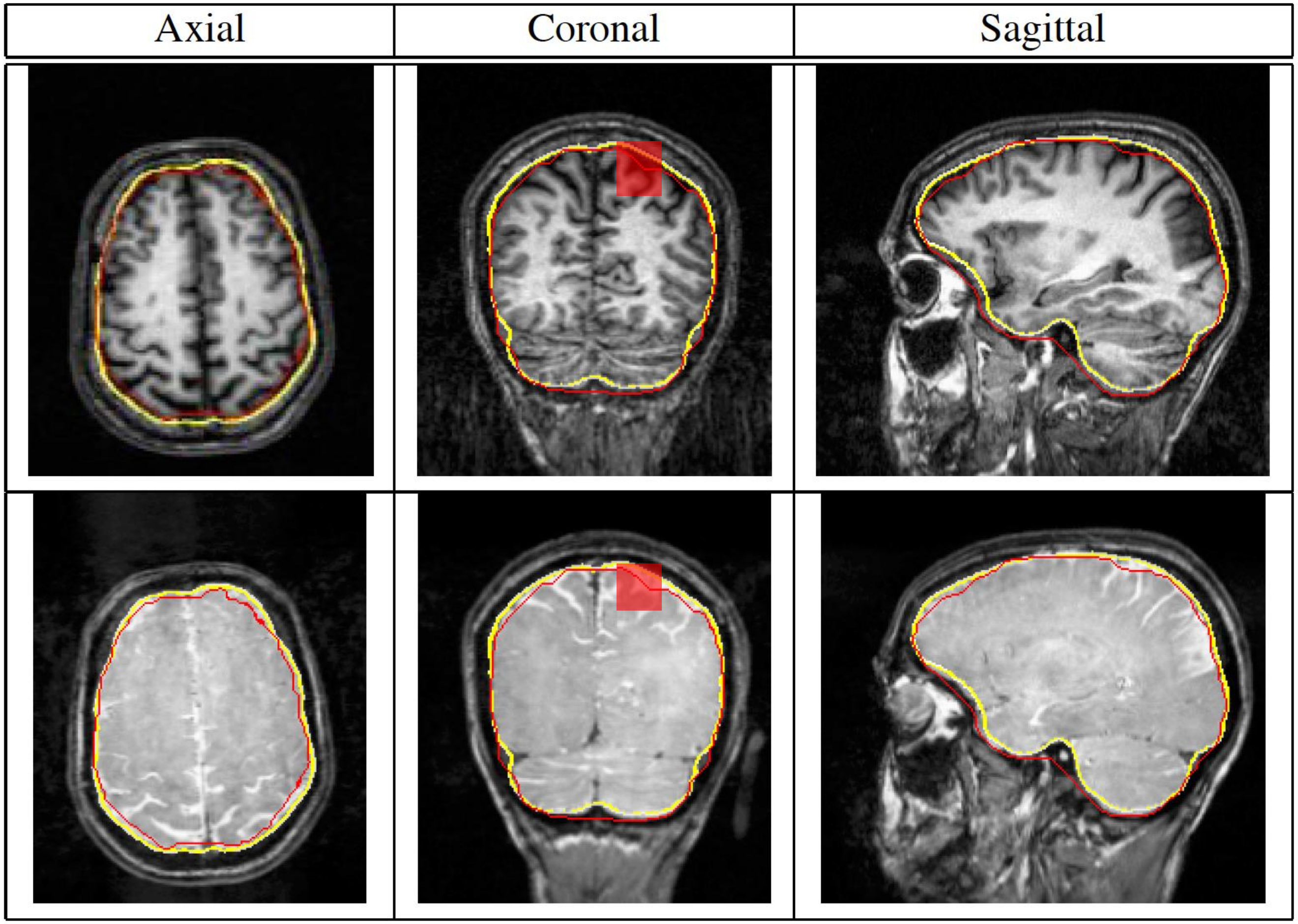
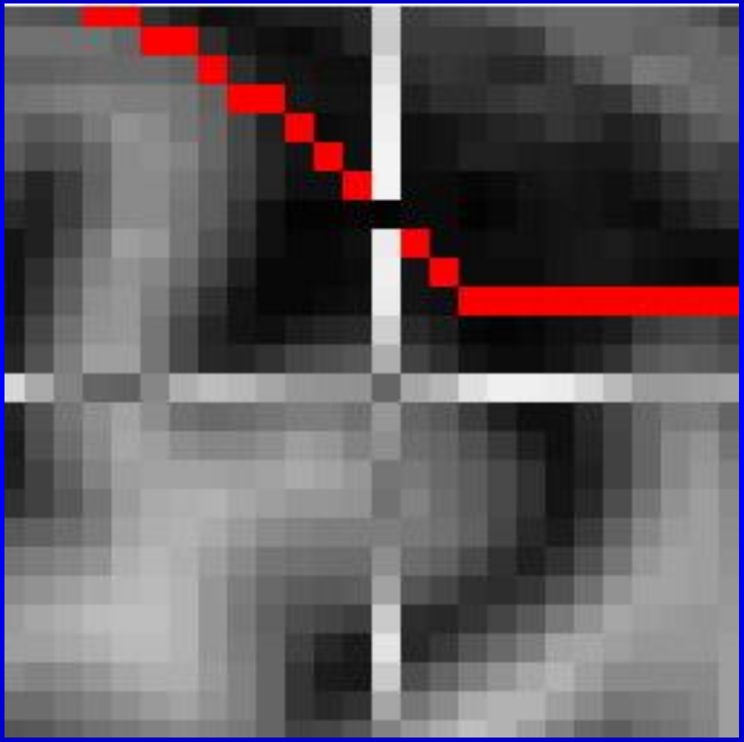
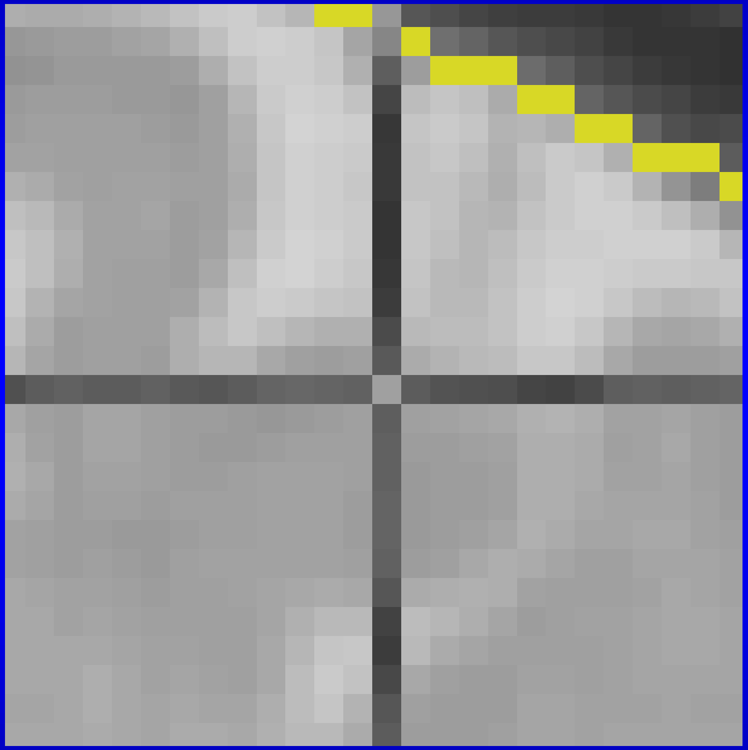


Figure 2.17: *Comparison of the inner skull segmentation results, using the bi-modal data set and Script 2.6.4 (border of ISS, yellow) or exclusively the T1 image and Script 2.6.5 (border of EISS, red).*

Influence of MI-based registration of T1/PD-MRI on inner skull segmentation



Result if only T1-MRI is available



Result if also well-registered PD-MRI is available

Thank you for your attention