CS 6640: IMAGE PROCESSING
Spring 2009
Test #1

Name: ________________________________
Student ID Number: ________________________________

Rules:

- Closed book.
- One page of notes (front and back).
- No calculators.

Hints:

- The term "describe" does not mean complete sentences and paragraphs or essays. If it's easier you may use simple bullets and meaningful phrases to answer such questions.
- If you split answers across pages (or on the backs of pages) make a clear note on the page where the question is posed to indicate you have done so. Clearly note the question number (and part) on the separate page.
- There are **four** questions for a total of 100 points.

1. **[25 pts.]** This question deals with histogram equalization using a analytically specified greyscale probability density function. Consider an image with the following greyscale PDF:

$$H(g) = \begin{cases} 4g & 0 \le g \le \frac{1}{2} \\ 4 - 4g & \frac{1}{2} \le g \le 1 \end{cases} .$$

   (a) Graph the PDF, $H(g)$ for the image.

   The graph looks like a triangle or tent function.

   (b) Derive an analytical expression for the greyscale transformation needed to equalize the histogram of this image.

   The greyscale transformation is the cumulative distribution. Thus, you integrate the PDF from $-\infty$ to g. We will use α as our variable for integration.

$$C(g) = \int_{-\infty}^{g} H(\alpha) d\alpha == \begin{cases} 0 & g < 0 \\ \int_0^g 4\alpha d\alpha = 2g^2 & 0 \le g \le \frac{1}{2} \\ \\ \frac{1}{2} + \int_{1/2}^{g} (4 - 4\alpha) d\alpha & \\ = \frac{1}{2} + 4g - 2g^2 - (4\frac{1}{2} - 2\frac{1}{4}) & \frac{1}{2} \le g \le 1 \\ = 4g - 2g^2 - 1 & \\ 0 & g > 1 \end{cases}$$

   (c) Graph the greyscale transformation needed to equalize the histogram of this image.

   The graph starts at $(0, 0)$, rises quadratically, like a parabola, and then at $1/2$, the slope starts decreasing, so that the slope is zero at 1.

   (d) Describe, very briefly, the effects of this greyscale transformation on the image. Which greyscale values undergo increased contrast and what greyscale values undergo decreased contrast?

   The degree of contrast enhancement (spreading out of greyscale values) is proportional to the derivative of C, or the magnitude of H. The, regions where the greyscales are near $1/2$ should see an increase in contrast. There will be decreases in contrast in dark or light regions.

2. **[20 pts.]** Prove (through equations) that the convolution of two Gaussians of different widths (standard deviations) is another Gaussian. I.e.

$$g_{\sigma_3}(x) = g_{\sigma_1}(x) \otimes g_{\sigma_2}(x)$$

   Give the equation for σ_3 in terms of σ_1 and σ_2.

   The equation for the Gaussian with standard deviation σ is:

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{x^2}{2\sigma^2}}$$

and the Fourier transform of this is

$$\mathbb{F}[g_\sigma] = G_\sigma(u) = e^{\pi^2\sigma^2u^2}$$

If we do our convolution in the Fourier domain we have:

$$G_{\sigma_1}(u)G_{\sigma_2}(u) = e^{\pi^2\sigma_1^2u^2}e^{\pi^2\sigma_2^2u^2} = e^{\pi^2(\sigma_1^2+\sigma_2^2)u^2} = e^{\pi^2\sigma_3^2u^2}$$

which is another Gaussian with $\sigma_3 = \sqrt{\sigma_1^2 + \sigma_2^2}$.

3. **[25 pts.]** Suppose we know that a bad telescope produces a distorted image $g(x, y) = h(x, y) \otimes f(x, y)$, where $f(x, y)$ is the correct image.

   (a) How might we go about finding out what $h(x, y)$ is?

   If we use a δ-function as input, then the output of the system is simply $h(x, y)$. To approximate this we could take a picture of some empty space that contains only a single, small object. Alernatively, we could take a picture of any scene of which we know the Fourier transform.

   (b) Suppose we know $h(x, y)$. How might we recover $f(x, y)$ from a given $g(x, y)$?

   $$f(x, y) = \mathbb{F}^{-1}\left[\frac{G(u, v)}{H(u, v)}\right]$$

   (c) From 3b, what problems do we expect to encounter and how might we get around it? (Hint: Think about how we approximately invert the diagonal matrix W associated with SVD).

   There could be divide-by-zero problems points in (u, v) where $|H(u, v)| \approx 0$. One solution would be to do what we did with the did with inverting the diagonal matrix for the SVD inverse. That is, the output F would be:

   $$F(u, v) = \begin{cases} G(u, v)/H(u, v) & |H(u, v)| > \epsilon \\ 0 & \text{otherwise} \end{cases},$$

   where ϵ is some small number. Another option, which gives a similar result is to include an extra term in the division. This would be:

   $$F(u, v) = \frac{G(u, v)}{H(u, v) + \epsilon}.$$

4. **[30 pts.]** This questions deals with the Fourier transform of a 1D, analytical function.

$$f(t) = \begin{cases} (1 + t)\cos\left(\frac{3}{2}\pi t\right) & -1 \le t \le 0 \\ (1 - t)\cos\left(\frac{3}{2}\pi t\right) & 0 \le t \le 1 \end{cases}.$$

(a) Make an approximate graph for the function $f(t)$.

The graph is cosine that falls off. One middle positive lobe, and two smaller negative lobes, one on each side (side lobes).

(b) Derive the Fourier transform $F(s)$ for this function.

Note that $f(t)$ is the multiplication of two functions. That is

$$f(t) = \mathrm{tri(t)} \cos\left(\frac{3}{2}\pi \mathrm{t}\right).$$

Thus, we can take advantage of the fact that f is a product in the time domain, which gives a convolution in the Fourier domain. The triangle function is the convolution of two rectangle functions, and the Fourier transform of the rect function is sinc. Thus, we have

$$\mathbb{F}[\mathrm{tri(t)}] = \mathrm{sinc^2(s)}.$$

For cosine we have

$$\mathbb{F}[\cos\left(\frac{3}{2}\pi t\right)] = \frac{1}{2}[\delta(s + \frac{3}{4}) + \delta(s - \frac{3}{4})].$$

Convolution with a translated δ-functional results in a shift in a function, thus, the Fourier transform is

$$\mathbb{F}[f(t)] = \mathbb{F}[\mathrm{tri(t)}] * \mathbb{F}[\cos\left(\frac{3}{2}\pi \mathrm{t}\right)] = \frac{1}{2}[\mathrm{sinc^2(s} + \frac{3}{4}) + \mathrm{sinc^2(s} - \frac{3}{4})]]$$

(c) Show or prove that the integral of $\int_{-\infty}^{\infty} f(t)dt = 0$ (Hint: easy in the Fourier domain).

There was a typo in this question that makes this part invalid. *Credit for this part of the question was given for any reasonable, valid* attempt *to solve this.*

The equation should have been

$$f(t) = \mathrm{tri(t)} \cos\left(2\pi \mathrm{t}\right).$$

In this case the Fourier transform would have been:

$$F(s) = \frac{1}{2}[\mathrm{sinc^2(s + 1) + sinc^2(s - 1)}].$$

In the Fourier domain, the integral of the function is capture in the value $F(0)$. The sinc function is $\sin(\pi s)/(\pi s)$, and because $\sin(\pi) = \sin(-\pi) = 0$, we know that $F(0) = 0$. You can also solve this problem by actually computing the integrals in the time domain, and using integration by parts.

(d) If you had to choose would you consider this function to be a low-pass, band-pass, or high-pass filter? Justify (briefly) your choice.

This filter (as specified in the original question) consists of two offset sinc functions, centered at $\pm 3\pi/4$, and thus it is a band-pass filter. This filter also follows the basic form (e.g. from the book) for band-pass filters, which is an pair offset low-pass filters.