Evaluation of Time Integration Schemes for the Generalized Interpolation Material Point Method

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Recurring Themes in Time Integration

Close coupling between space and time

Low order is better

Linear error theories not applicable

Behavior drastically different for large deformation

So what's really going on?

Manufactured solutions measure effect of time integration on

- Accuracy
- Stability

Compare USF and USL

Update Stress First

$$\nabla \mathbf{v}_p^n = \sum_i \mathbf{v}_i \mathbf{G}_{ip}$$

$$\mathbf{F}_p^{n+1} = \mathbf{F}_p^n + \nabla \mathbf{v}_p^n \cdot \mathbf{F}_p^n \Delta t$$

$$\mathbf{a}_i\left(\boldsymbol{\sigma}\left(\mathbf{F}_p^{n+1}\right)\right)$$

Update Stress Last

$$\mathbf{a}_i\left(\boldsymbol{\sigma}\left(\mathbf{F}_p^{n}\right)\right)$$

$$\nabla \mathbf{v}_p^{n+1} = \sum_i (\mathbf{v}_i + \mathbf{a}_i \Delta t) \mathbf{G}_{ip}$$

$$\mathbf{F}_p^{n+1} = \mathbf{F}_p^n + \nabla \mathbf{v}_p^{n+1} \cdot \mathbf{F}_p^n \Delta t$$

Compare Centered-Difference and USL

$$\mathbf{v}_i^{n-1/2} = \frac{\sum \mathbf{v}_p^{n-1/2} m_p \mathbf{S}_{ip}}{\sum m_p \mathbf{S}_{ip}}$$

$$\mathbf{v}_i = \frac{\sum \mathbf{v}_p^{n} m_p \mathbf{S}_{ip}}{\sum m_p \mathbf{S}_{ip}}$$

$$\mathbf{v}_p^{n+1/2} = \mathbf{v}_p^{n-1/2} + \mathbf{a}_p^n \Delta t$$

$$\mathbf{v}_p^{n+1} = \mathbf{v}_p^{n} + \mathbf{a}_p \Delta t$$

$$\mathbf{x}_p^{n+1} = \mathbf{x}_p^{n} + \mathbf{v}_p^{n+1/2} \Delta t$$

$$\mathbf{x}_p^{n+1} = \mathbf{x}_p^{n} + \mathbf{v}_p \Delta t$$

$$\mathbf{F}_p^{n+1} = \mathbf{F}_p^{n} + \nabla \mathbf{v}_p^{n+1/2} \cdot \mathbf{F}_p^{n} \Delta t$$

$$\mathbf{F}_p^{n+1} = \mathbf{F}_p^{n} + \nabla \mathbf{v}_p^{n+1} \cdot \mathbf{F}_p^{n} \Delta t$$

Initialization to a Negative Half Time Step

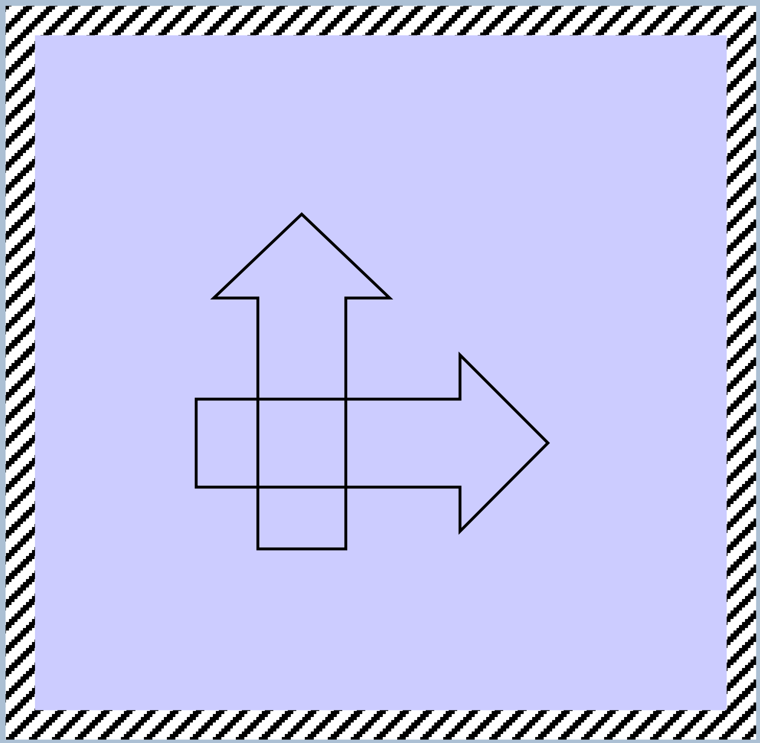
If you know the answer: $$v_p = v(t = -k/2)$$

Use data at time = 0: $$v_p^{-1/2} = v_p^0 - \frac{\Delta t}{2} \boxed{a_p^0}$$

The easy way:
if first time step then $$a_i^0 = \frac{1}{2} a_i^0$$

Axis-Aligned Displacement in a Unit Square

$$u = \begin{pmatrix} A\sin(\pi X)\cos(c\pi t) \\ A\sin(\pi Y)\sin(c\pi t) \\ 0 \end{pmatrix}$$



Functions of coordinate directions only

Corners and edges of GIMP particles remain aligned

Sliding boundaries – zero normal velocity at surface.

Axis-Aligned Displacement – cont.

Diagonal terms only:

$$F = \begin{bmatrix} 1 + A\pi\cos(\pi X)\cos(c\pi t) & 0 & 0 \\ 0 & 1 + A\pi\cos(\pi Y)\sin(c\pi t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Stress

$$\mathbf{P} = \lambda \ln(J)\mathbf{F}^{-1} + \mu\mathbf{F}^{-1}\left(\mathbf{F}\mathbf{F}^{T} - \mathbf{I}\right)$$

Momentum

$$\nabla \cdot \mathbf{P} + \rho_0 \mathbf{b} = \rho_0 \mathbf{a}$$

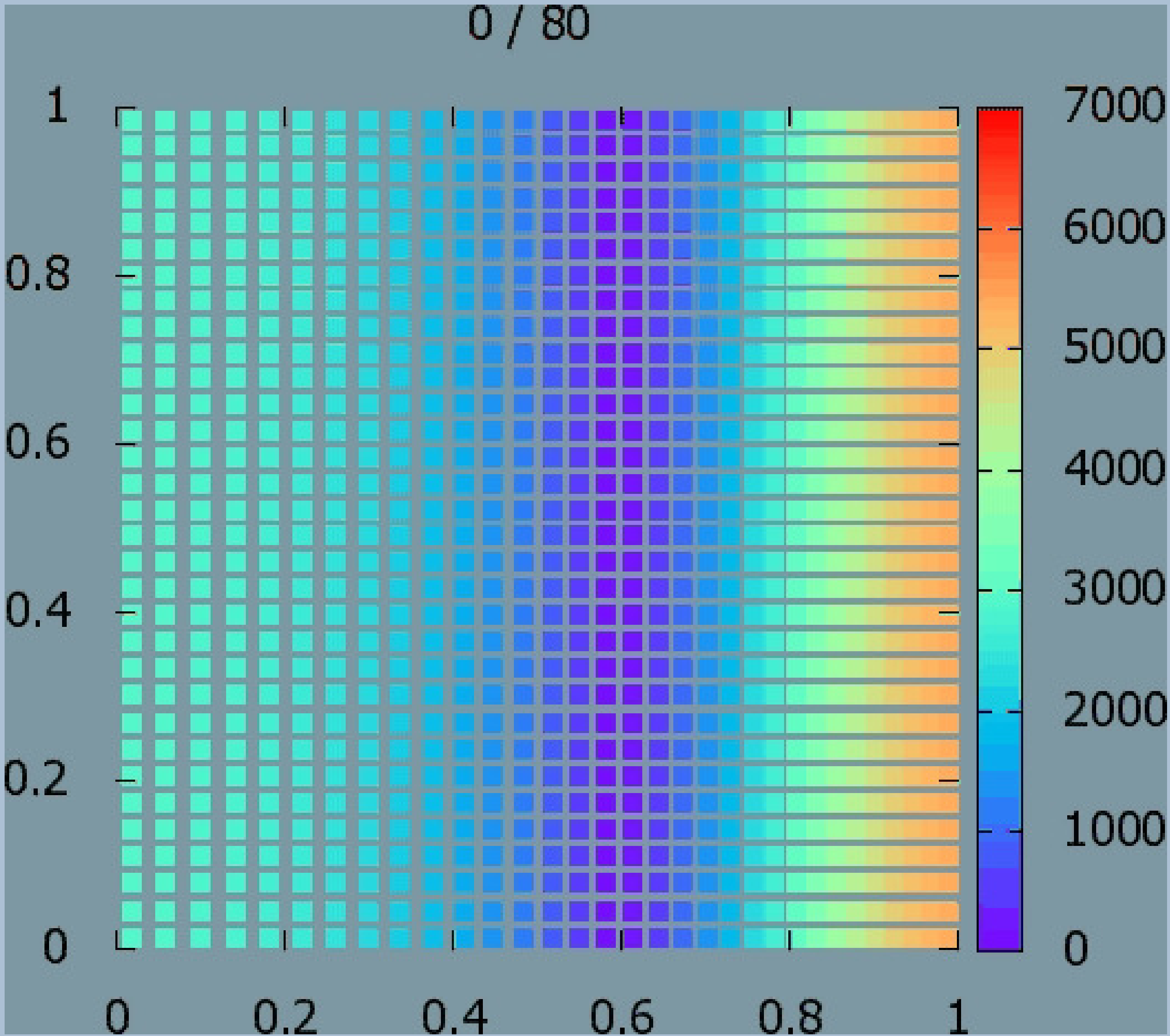
Solve for body force from momentum:

$$\mathbf{b} = \frac{\pi^2}{\rho_0}\begin{pmatrix} u_X\left[\lambda\left(1 - \ln(F_{XX}F_{YY})\right)F_{XX}^{-2} + \mu\left(1 + F_{XX}^{-2}\right) - E\right] \\ u_Y\left[\lambda\left(1 - \ln(F_{XX}F_{YY})\right)F_{YY}^{-2} + \mu\left(1 + F_{YY}^{-2}\right) - E\right] \\ 0 \end{pmatrix}$$

Axis-Aligned Displacement

Von Mises Stress

Straight rows and columns only with the right answer



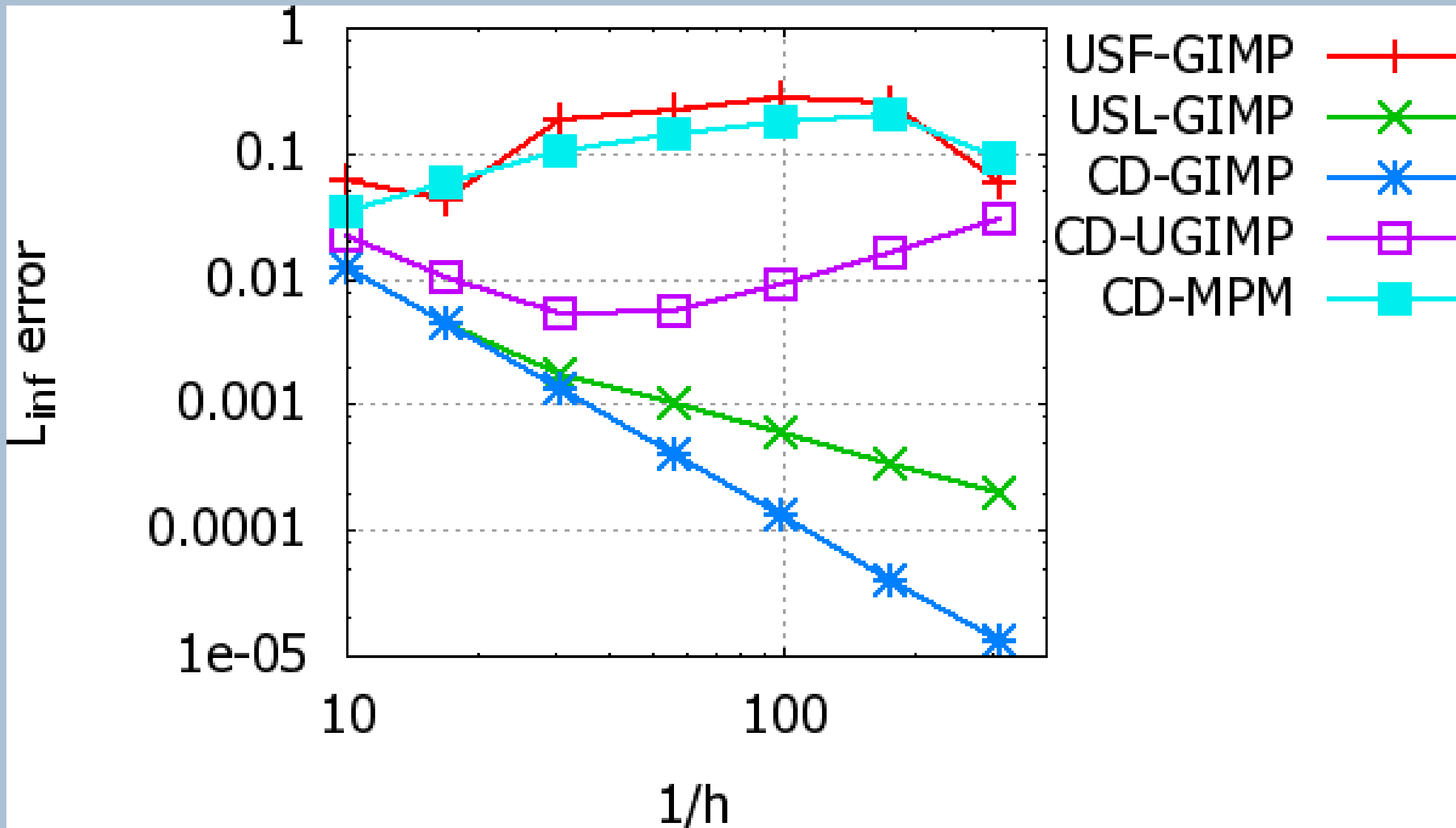
Definition of Error at a Particle

$$\delta = \left(x_p^n - X_p^n\right) - u(X_p^n, t^n)_{EXACT}$$

For a smooth problem in space and time check all particles and all time steps:

$$L_\infty = \max\left(\delta\right)$$

Spatial Convergence



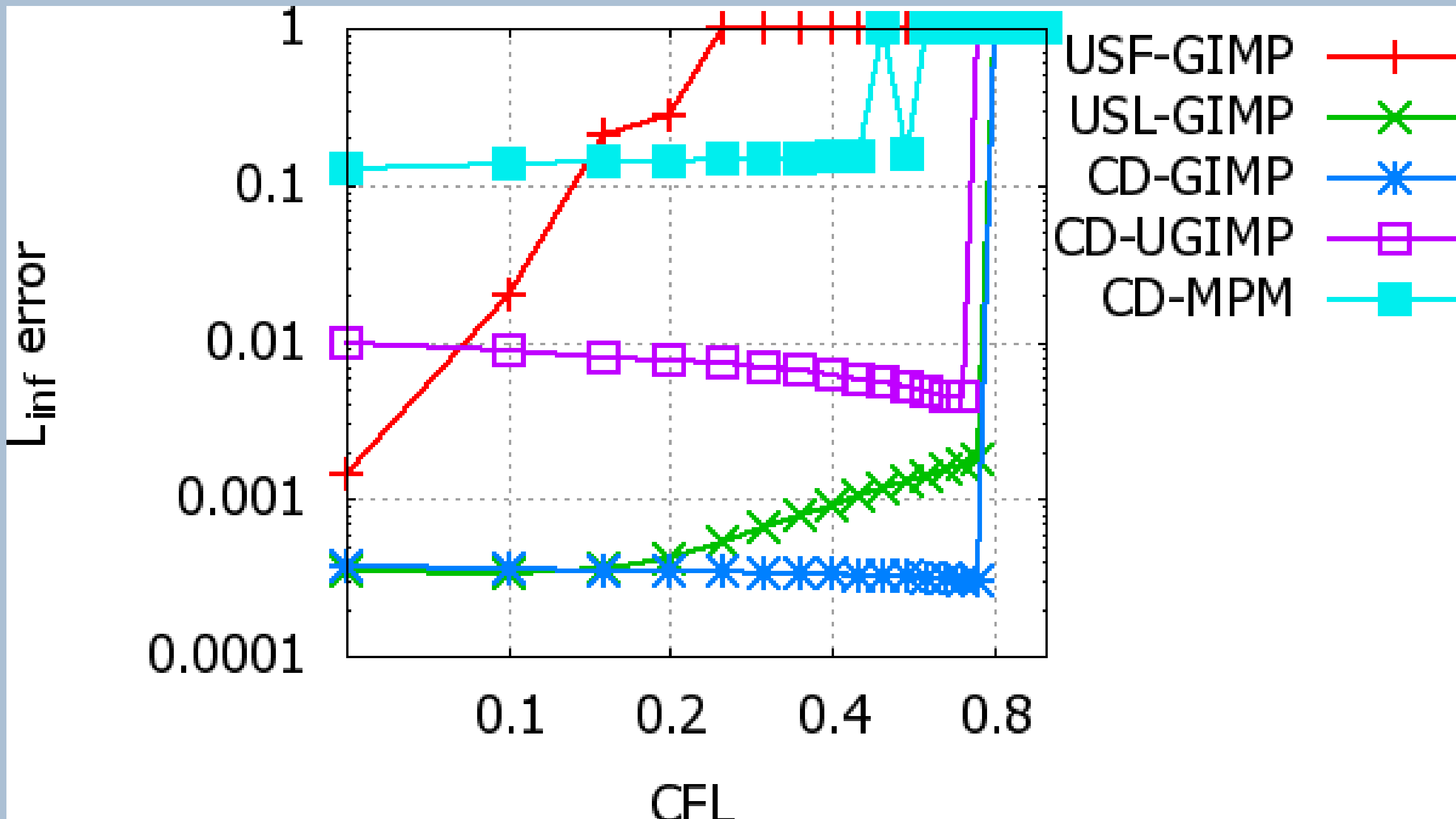
CD-GIMP is 2nd order – the initialization shortcut works

USL changes to 1st order – effect of half step initialization

Minor change to UGIMP causes large error

USF and MPM visually OK but poor accuracy

Temporal Convergence



Most methods display zero temporal convergence until stability is lost, even though CD-GIMP is formally 2nd order in time.

USL loses one spatial order, and gains one temporal – sum of 2?

We conclude that spatial error dominates temporal error such that reduced CFL has no benefit.

GIMP Convergence Order

Hypothesis

GIMP is <u>second order</u> in space when:

1. Problem is smooth in space and time
2. Particle edges are aligned
3. Material boundary is accurately represented

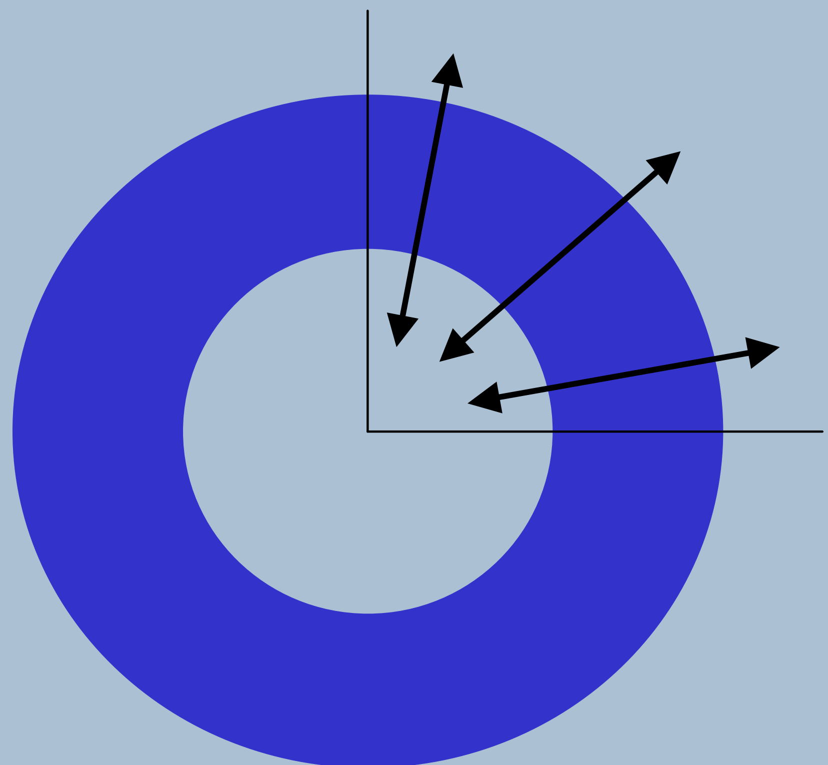
Therefore the 2D code is verified.

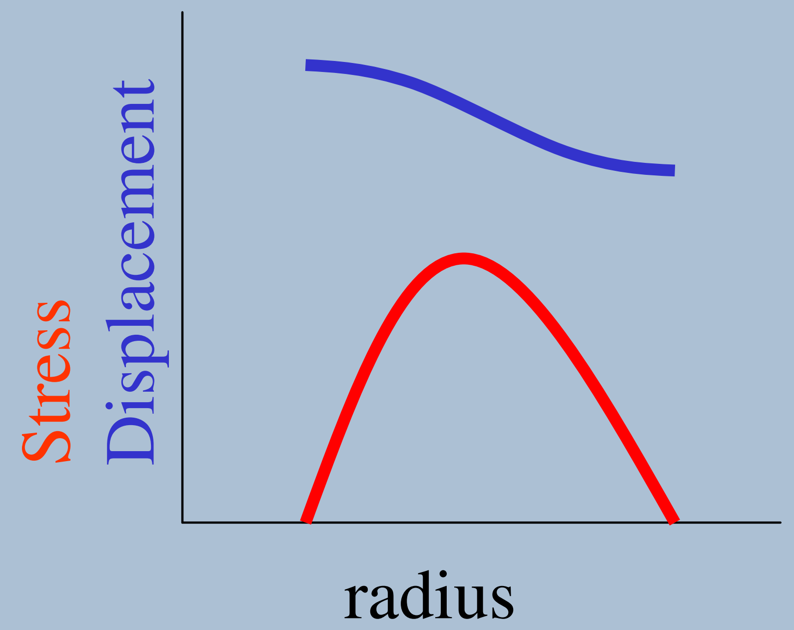
Expanding Ring

Free surfaces with implied zero normal stress

GIMP particle edges not aligned

More general and representative





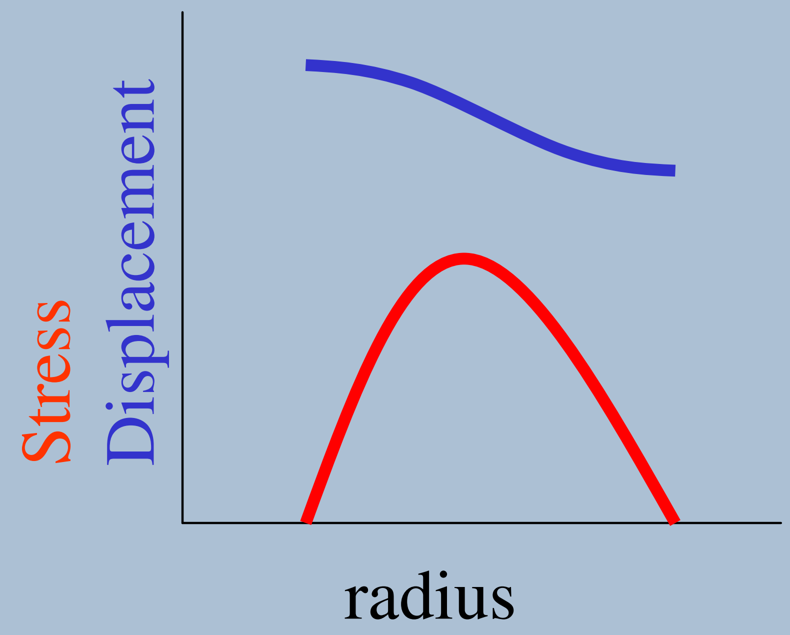
Expanding Ring – Displacement

Radial Symmetry $$u(R,t) = T(t)\left[c_3 R^3 + c_2 R^2 + c_1 R\right]$$

Capital "R" is radius in reference configuration.

X and Y Displacement Components:

$$\mathbf{u} = \begin{bmatrix} T\left[c_3 R^3 + c_2 R^2 + c_1 R\right]\cos(\theta) \\ T\left[c_3 R^3 + c_2 R^2 + c_1 R\right]\sin(\theta) \\ 0 \end{bmatrix}$$



Expanding Ring: Cartesian Coordinates

Gradient Operators in terms of R and θ:

$$\frac{\partial f(R,\theta)}{\partial X} = \frac{\partial f}{\partial R}\cos(\theta) - \frac{\partial f}{\partial \theta}\frac{\sin(\theta)}{R}$$

$$\frac{\partial f(R,\theta)}{\partial Y} = \frac{\partial f}{\partial R}\sin(\theta) + \frac{\partial f}{\partial \theta}\frac{\cos(\theta)}{R}$$

$$\mathbf{F} = \mathbf{I} + \begin{bmatrix} \frac{\partial u_X}{\partial X} & \frac{\partial u_X}{\partial Y} & 0 \\ \frac{\partial u_Y}{\partial X} & \frac{\partial u_Y}{\partial Y} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Stress with zero Poisson's ratio

$$\mathbf{P} = \mu \mathbf{F}^{-1}\left(\mathbf{F}\mathbf{F}^T - \mathbf{I}\right)$$

Now we let Maple do the hard part . . .

Stress Matrix

$P :=$

$$\Big[\frac{1}{2} E\, T\, (T^2 \cos(H)^2 c2\, R\, c1^2 + 3 T^2 c2^2 R^2 \cos(H)^2 c1 + 2 T^2 \cos(H)^2 c3\, R^2 c1^2 + 2 T \cos(H)^2 c2\, R\, c1 + 4 T \cos(H)^2 c3\, R^2 c1$$
$$+ 10 T^2 c3\, R^3 \cos(H)^2 c2\, c1 + 8 T^2 c3^2 R^4 \cos(H)^2 c1 + 13 T^2 c3^2 R^5 \cos(H)^2 c2 + 9 T^2 c3\, R^4 \cos(H)^2 c2^2 + 10 T \cos(H)^2 c3\, R^3 c2$$
$$+ 5 T c2^2 R^2 + 7 T c3^2 R^4 + T^2 c1^3 + 3 T c1^2 + 2 T^2 c2^3 R^3 + 3 T^2 c3^3 R^6 + 8 T c2\, R\, c1 + 10 T c3\, R^2 c1 + 12 T c3\, R^3 c2 + 7 T^2 c3\, R^4 c2^2$$
$$+ 5 T^2 c3\, R^2 c1^2 + 7 T^2 c3^2 R^4 c1 + 8 T^2 c3^2 R^5 c2 + 5 T^2 c2^2 R^2 c1 + 4 T^2 c2\, R\, c1^2 + 12 T^2 c3\, R^3 c2\, c1 + 6 T^2 c3^3 R^6 \cos(H)^2 + 2 c3\, R^2 + 2 c1$$
$$+ 2 T^2 c2^3 R^3 \cos(H)^2 + 2 c2\, R + 3 T \cos(H)^2 c2^2 R^2 + 8 T \cos(H)^2 c3^2 R^4 + 2 \cos(H)^2 c2\, R + 4 \cos(H)^2 c3\, R^2) \Big/ ($$
$$1 + 3 T^2 c3^2 R^4 + 5 T^2 c3\, R^3 c2 + 4 T^2 c3\, R^2 c1 + 3 T^2 c2\, R\, c1 + T^2 c1^2 + 2 T c1 + 2 T^2 c2^2 R^2 + 3 T c2\, R + 4 T c3\, R^2), \frac{1}{2} E\, T \cos(H) \sin(H)\, R$$
$$(2 c3\, R + c2)\, (2 + 3 T^2 c3^2 R^4 + 5 T^2 c3\, R^3 c2 + 4 T^2 c3\, R^2 c1 + 3 T^2 c2\, R\, c1 + T^2 c1^2 + 2 T c1 + 2 T^2 c2^2 R^2 + 3 T c2\, R + 4 T c3\, R^2) \Big/ ($$
$$1 + 3 T^2 c3^2 R^4 + 5 T^2 c3\, R^3 c2 + 4 T^2 c3\, R^2 c1 + 3 T^2 c2\, R\, c1 + T^2 c1^2 + 2 T c1 + 2 T^2 c2^2 R^2 + 3 T c2\, R + 4 T c3\, R^2), 0\Big]$$

$$\Big[\frac{1}{2} E\, T \cos(H) \sin(H)\, R\, (2 c3\, R + c2)$$
$$(2 + 3 T^2 c3^2 R^4 + 5 T^2 c3\, R^3 c2 + 4 T^2 c3\, R^2 c1 + 3 T^2 c2\, R\, c1 + T^2 c1^2 + 2 T c1 + 2 T^2 c2^2 R^2 + 3 T c2\, R + 4 T c3\, R^2) \Big/ ($$
$$1 + 3 T^2 c3^2 R^4 + 5 T^2 c3\, R^3 c2 + 4 T^2 c3\, R^2 c1 + 3 T^2 c2\, R\, c1 + T^2 c1^2 + 2 T c1 + 2 T^2 c2^2 R^2 + 3 T c2\, R + 4 T c3\, R^2), -\frac{1}{2} T\, E\, ($$
$$T^2 \cos(H)^2 c2\, R\, c1^2 + 3 T^2 c2^2 R^2 \cos(H)^2 c1 + 2 T^2 \cos(H)^2 c3\, R^2 c1^2 + 2 T \cos(H)^2 c2\, R\, c1 + 4 T \cos(H)^2 c3\, R^2 c1$$
$$+ 10 T^2 c3\, R^3 \cos(H)^2 c2\, c1 + 8 T^2 c3^2 R^4 \cos(H)^2 c1 + 13 T^2 c3^2 R^5 \cos(H)^2 c2 + 9 T^2 c3\, R^4 \cos(H)^2 c2^2 + 10 T \cos(H)^2 c3\, R^3 c2$$
$$- 8 T c2^2 R^2 - 15 T c3^2 R^4 - T^2 c1^3 - 3 T c1^2 - 4 T^2 c2^3 R^3 - 9 T^2 c3^3 R^6 - 10 T c2\, R\, c1 - 14 T c3\, R^2 c1 - 22 T c3\, R^3 c2 - 16 T^2 c3\, R^4 c2^2$$
$$- 7 T^2 c3\, R^2 c1^2 - 15 T^2 c3^2 R^4 c1 - 21 T^2 c3^2 R^5 c2 - 8 T^2 c2^2 R^2 c1 - 5 T^2 c2\, R\, c1^2 - 22 T^2 c3\, R^3 c2\, c1 + 6 T^2 c3^3 R^6 \cos(H)^2 - 6 c3\, R^2$$
$$- 2 c1 + 2 T^2 c2^3 R^3 \cos(H)^2 - 4 c2\, R + 3 T \cos(H)^2 c2^2 R^2 + 8 T \cos(H)^2 c3^2 R^4 + 2 \cos(H)^2 c2\, R + 4 \cos(H)^2 c3\, R^2) \Big/ ($$
$$1 + 3 T^2 c3^2 R^4 + 5 T^2 c3\, R^3 c2 + 4 T^2 c3\, R^2 c1 + 3 T^2 c2\, R\, c1 + T^2 c1^2 + 2 T c1 + 2 T^2 c2^2 R^2 + 3 T c2\, R + 4 T c3\, R^2), 0\Big]$$

$$[0, 0, 0]$$

Find c_1, c_2, and c_3 by rotating the stress matrix

Rotation Matrix Q: $\mathbf{Q} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\mathbf{P'} = \mathbf{QPQ}^T = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\alpha(R_O) = 0$$

$$\alpha(R_I) = 0$$

$$u(R_O) = T$$

Maple finds a simple answer:

$$\{ c3 = 2 \frac{1}{Ro^2 (3\,Ri - Ro)},\ c2 = -3 \frac{Ro + Ri}{Ro^2 (3\,Ri - Ro)},\ c1 = 6 \frac{Ri}{(3\,Ri - Ro)\,Ro} \},$$

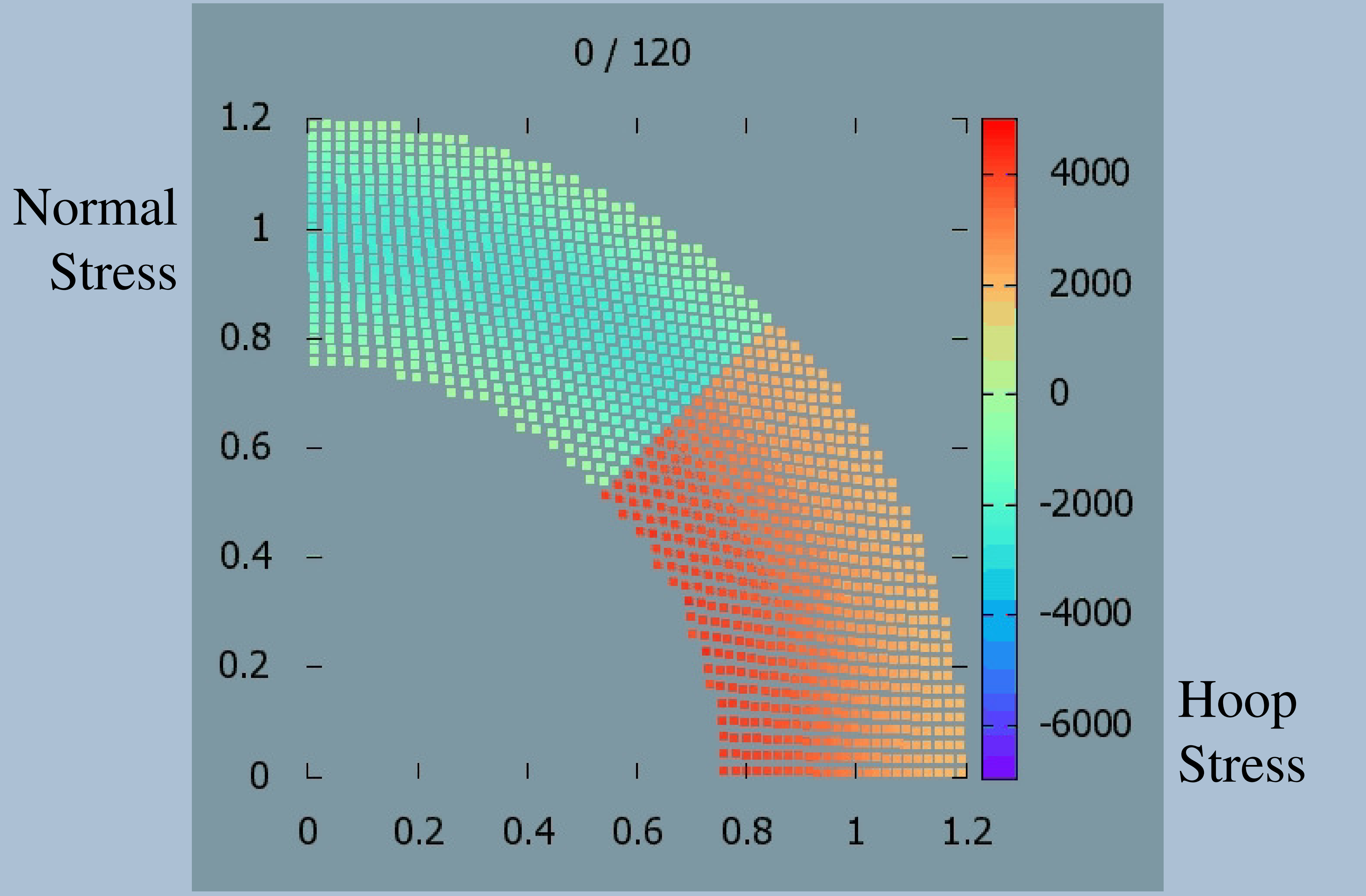
Solve for Body Force

$$\nabla \cdot \mathbf{P} + \rho_0 \mathbf{b} = \rho_0 \mathbf{a} \quad \Longrightarrow \quad \mathbf{b} = \frac{\partial^2 \mathbf{u}}{\partial t^2} - \frac{1}{\rho_0} \nabla \cdot \mathbf{P}$$

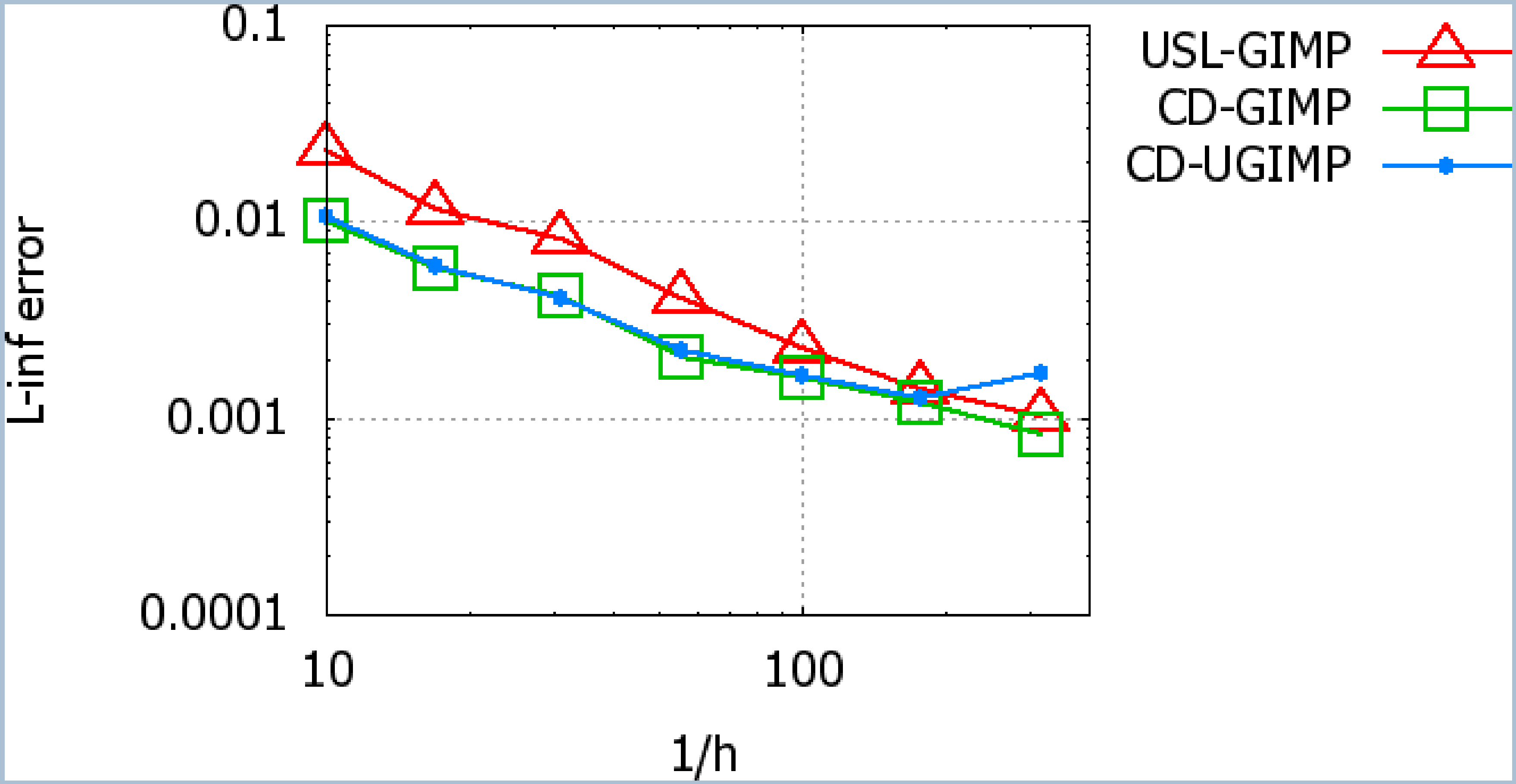
$$T(t) = A \cos\left(\sqrt{\frac{E}{\rho_0}} \, \pi \, \mathrm{t} \right)$$

Maple generates C-compatible code for b

Expanding Ring Results



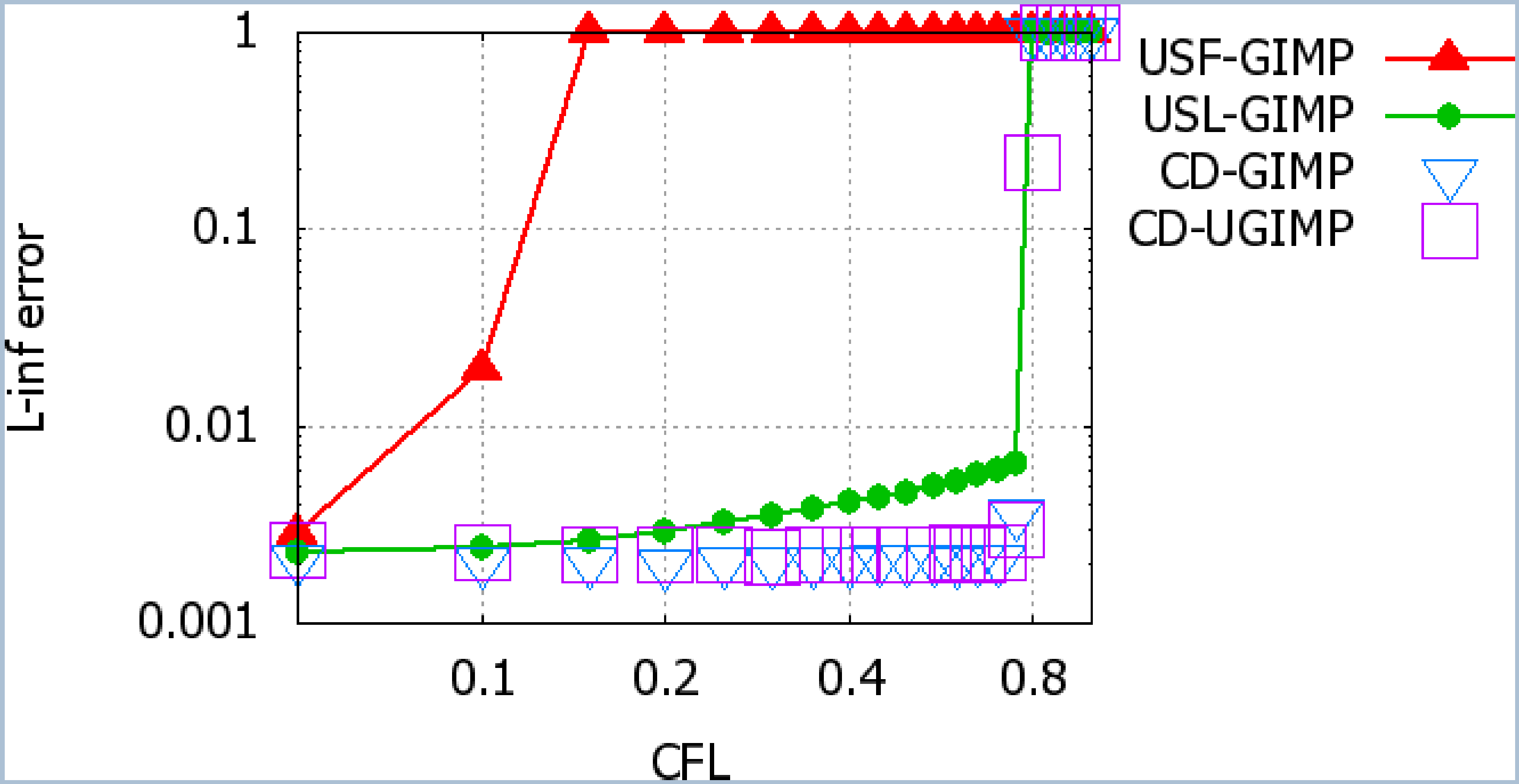
Ring – Spatial Convergence



The miss-alignment of particles and the stair-stepped surface now dominate the error.

UGIMP gives up only for the highest resolutions.

Ring – Temporal Convergence



UGIMP and GIMP nearly same accuracy

Same trends otherwise: little temporal convergence

Conclusions

The Method of Manufactured solutions allows order of accuracy to be demonstrated for realistic large deformation problems.

The CD-GIMP combination is significantly better than choices involving UGIMP, MPM, USF, and USL.

Formal temporal orders of accuracy are usually not observed in real solutions because spatial error dominates temporal.

CD-GIMP can be 2nd order if problem is smooth, particle edges are aligned, and surface is well-represented. Convergence drops to 1st order for non-aligned particles.

Thanks to Mike Steffen, Mike Kirby , LeThuy Tran, and Martin Berzins, as well as DOE grant W-7405-ENG-48.

Expanding Disk: C code from Maple

```
t1 = pi*pi;
t3 = 1/rho;
t4 = t1*E*t3;
t5 = R*R;
t6 = t5*R;
t8 = c2*t5;
t9 = c1*R;
t11 = T*(c3*t6+t8+t9);
t12 = cos(H);
t17 = T*T;
t18 = t17*T;
t19 = c2*c2;
t20 = t19*t19;
t27 = c3*c3;
t31 = T*c2;
t34 = t17*c2;
t35 = c1*c1;
t39 = t35*c1;
t42 = t27*t27;
t44 = t5*t5;
t48 = t19*c2;
```

```
t52 = t27*c3;
t54 = t44*R;
t57 = t18*t27;
t61 = t18*t52;
t66 = t18*c3;
t77 = t5*c1;
t80 =
12.0*t18*t20*t6+19.0*T*t19*R+6
8.0*T*t27*t6+12.0*t31*c1+9.0*t
34*t35+3.0*t18*c2*t39+72.0*t18
*t42*t44*t6+24.0*t17*t48*t5+12
0.0*t17*t52*t54+191.0*t57*
t54*t19+195.0*t61*t44*t5*c2+8.
0*t66*R*t39+56.0*t57*t6*t35+80
.0*t66*t44*t48+24.0
*t18*t48*t77;
t82 = R*t35;
t88 = t17*c3;
t89 = t6*t19;
t94 = t17*t27;
t98 = t44*c2;
t101 = t17*t19;
t104 = T*c3;
```

```
t124 =
15.0*t18*t19*t82+120.0*t61*t54*c1+1
30.0*t88*t89+24.0*t88*t82+112.0*t94
*t6*c1+221.0*t94*t98+30.0*t101*t9+3
2.0*t104*t9+76.0*t104*t8+6.0*c2+122
.0*
t88*t8*c1+61.0*t66*t8*t35+130.0*t66
*t89*c1+221.0*t57*t98*c1+16.0*c3*R;
t125 = t80+t124;
t127 = t104*t5;
t129 = t31*R;
t131 = T*c1;
t151 =
1/(3.0*t127+2.0*t129+1.0+t131)/(1.0
+3.0*t94*t44+5.0*t88*t6*c2+4.0*
t88*t77+3.0*t34*t9+t17*t35+2.0*t131
+2.0*t101*t5+3.0*t129+4.0*t127);
t156 = sin(H);
b[0] = -t4*t11*t12-
t3*E*T*t12*t125*t151/2.0;
b[1] = -t4*t11*t156-
t3*t125*E*T*t156*t151/2.0;
b[2] = 0.0;
```