Verification of GIMP with Manufactured Solutions

Philip Wallstedt – philip.wallstedt@utah.edu
Jim Guilkey – james.guilkey@utah.edu

C-SAFE
UNIVERSITY OF UTAH

THE UNIVERSITY OF UTAH

Verification: Necessary Versus Sufficient

Eyeball Norms – no obvious error

- Not predictive: you already know the answer

Symmetry – some coding mistakes exposed

- Many mistakes are symmetric

Compare to existing code (Finite Element)

- Existing code solves different problems
- Existing code has unverified accuracy
- When differences are found, are they errors or not?

Experimental results – scattered data shows same trends

- Lack of data
- Differences don't show what's wrong with code

Known Solutions to PDE's

- No solutions for large deformation

We need a better way: the Method of Manufactured Solutions (MMS)

Recently proposed as ASME standard

> "V&V 10 - 2006 Guide for Verification and Validation in Computational Solid Mechanics"

Sufficient, not just necessary, if we test all modes:

- Boundary conditions
- Non-square cells and particles
- Time integration algorithms
- Shape functions

Each mode must be tested, but not all in the same test.
Once a mode has "passed", then further testing not needed.

Rate of convergence is very sensitive to errors and can be applied to individual pieces of a method

Displacement error compares current config to reference.

$$\delta_u = \left(x_p - X_p\right) - u(X)_{EXACT}$$

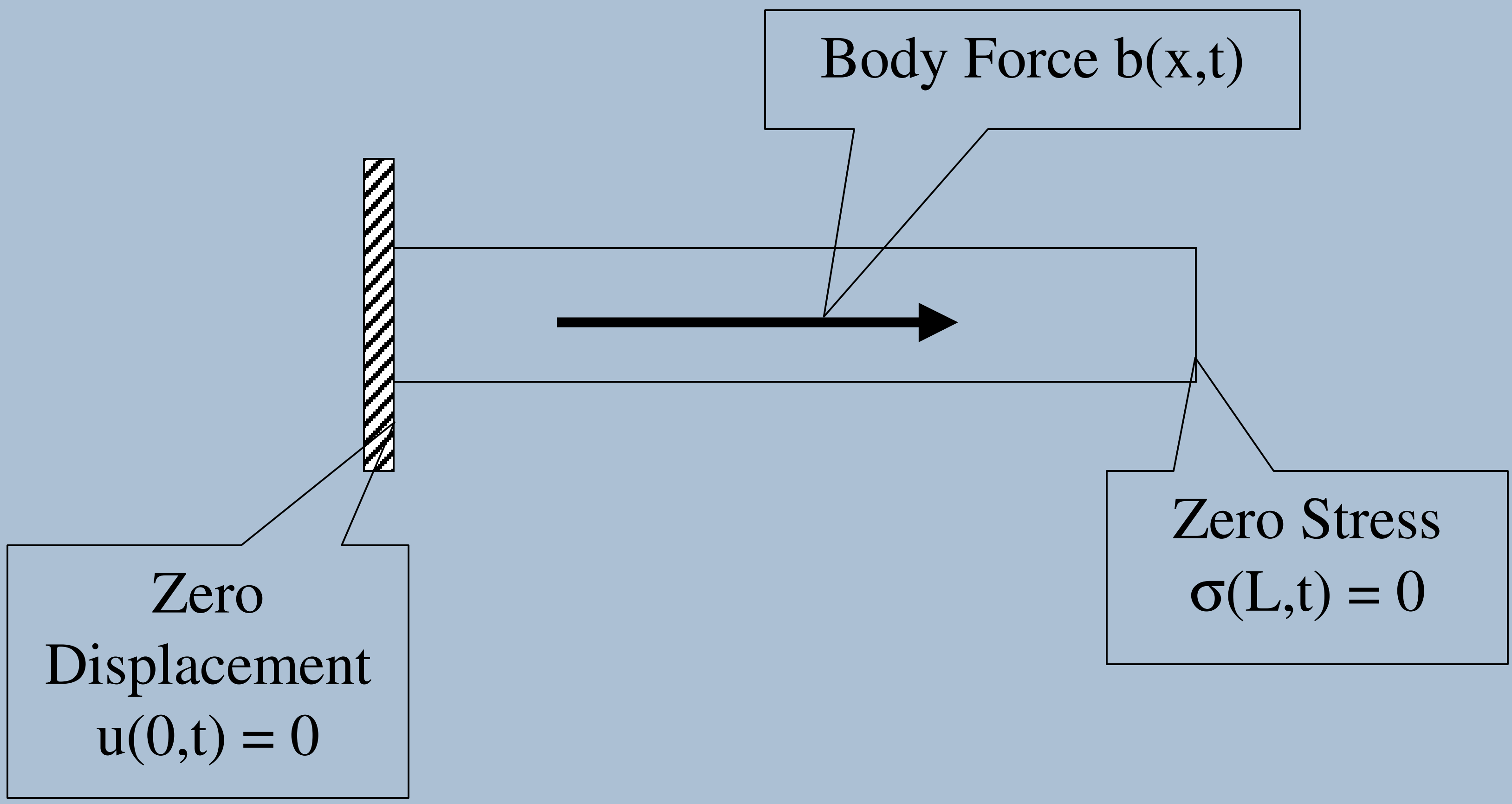
Average error

$$L_1 = \max\left(\frac{\sum_p \delta_p}{N}\right)$$

Worst Error

$$L_\infty = \max\left(\delta_p\right)$$

Body Force on a 1D Bar



Body Force on a 1D Bar

Given

Momentum $$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a}$$

Neo-Hookean Constitutive Model $$\boldsymbol{\sigma} = \frac{\lambda}{J} \ln J \mathbf{I} + \frac{\mu}{J}\left(\mathbf{F}\mathbf{F}^{\mathrm{T}} - \mathbf{I}\right)$$

Constitutive Model with assumptions: 1D, Poisson = 0 $$\boldsymbol{\sigma} = \frac{E}{2}\left(\mathbf{F} - \frac{1}{\mathbf{F}}\right)$$

Find displacement u(x) – in general this cannot be done.

Start with the answer and reformulate backwards

Given Displacement $\mathbf{u}(\mathbf{X})$

1D Neo-Hookean with Poisson's ratio = 0

$$\mathbf{P} = \frac{E}{2}\left(\mathbf{F} - \frac{1}{\mathbf{F}}\right)$$

Momentum

$$\nabla \cdot \mathbf{P} + \rho_0 \mathbf{b} = \rho_0 \mathbf{a}$$

Solve for Gravity

$$\mathbf{b} = \mathbf{a} - \frac{1}{\rho_0} \nabla \cdot \mathbf{P}$$

Now we just take derivatives . . .

What answer (displacement field) do we start with?

The chosen displacement field(s) must:

- exercise all features of the code (large deformation, translation, rotation, Dirichlet and Neumann boundaries
- be “smooth enough” – sufficiently differentiable in time and space
- Conform to assumptions made by the method. For GIMP this means zero normal stress at material boundaries.

For the 1D rod a parabolic form should work:

$$u = \left(c_0 + c_1 X + c_2 X^2\right) A(t)$$

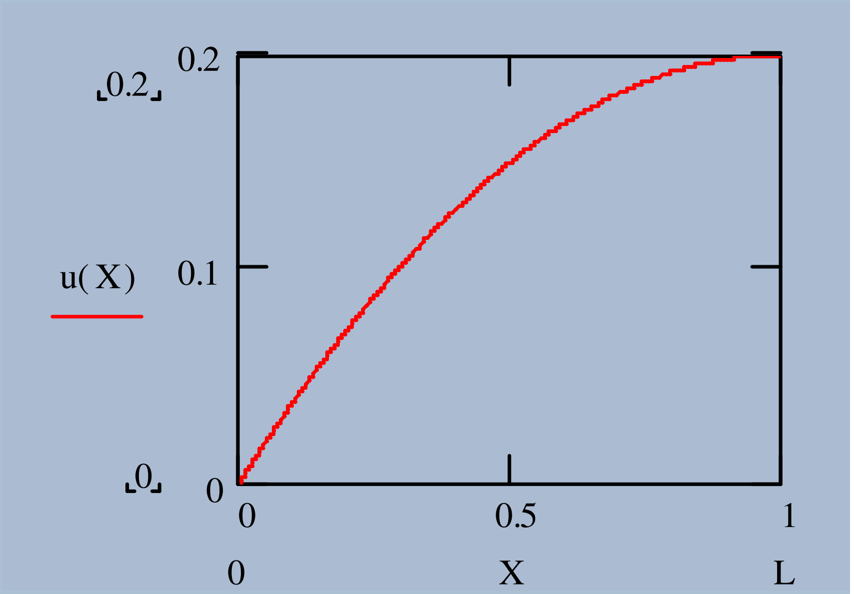
Constants for the 1D bar

Zero displacement at X = 0

$$0 = \left(c_0 + c_1 0 + c_2 0^2\right) A(t)$$

Scale displacement at L

$$A(t) = \left(c_0 + c_1 L + c_2 L^2\right) A(t)$$



Zero stress at X = L

$$P(L) = 0 = \frac{E}{2}\left(F - \frac{1}{F}\right) = \frac{E}{2}\left(1 + \left(c_1 + 2c_2 L\right)A(t) - \frac{1}{1 + \left(c_1 + 2c_2 L\right)A(t)}\right)$$

$$u = \frac{X(2L - X)}{L^2} A(t)$$

Choose a convenient time function A(t)

Trig function:

- Easy to differentiate
- Stays close to un-deformed shape
- Tests ability to preserve energy
- Can be made self-similar in time – same number of time steps per period, regardless of problem stiffness.

But other functions work just fine, provided:

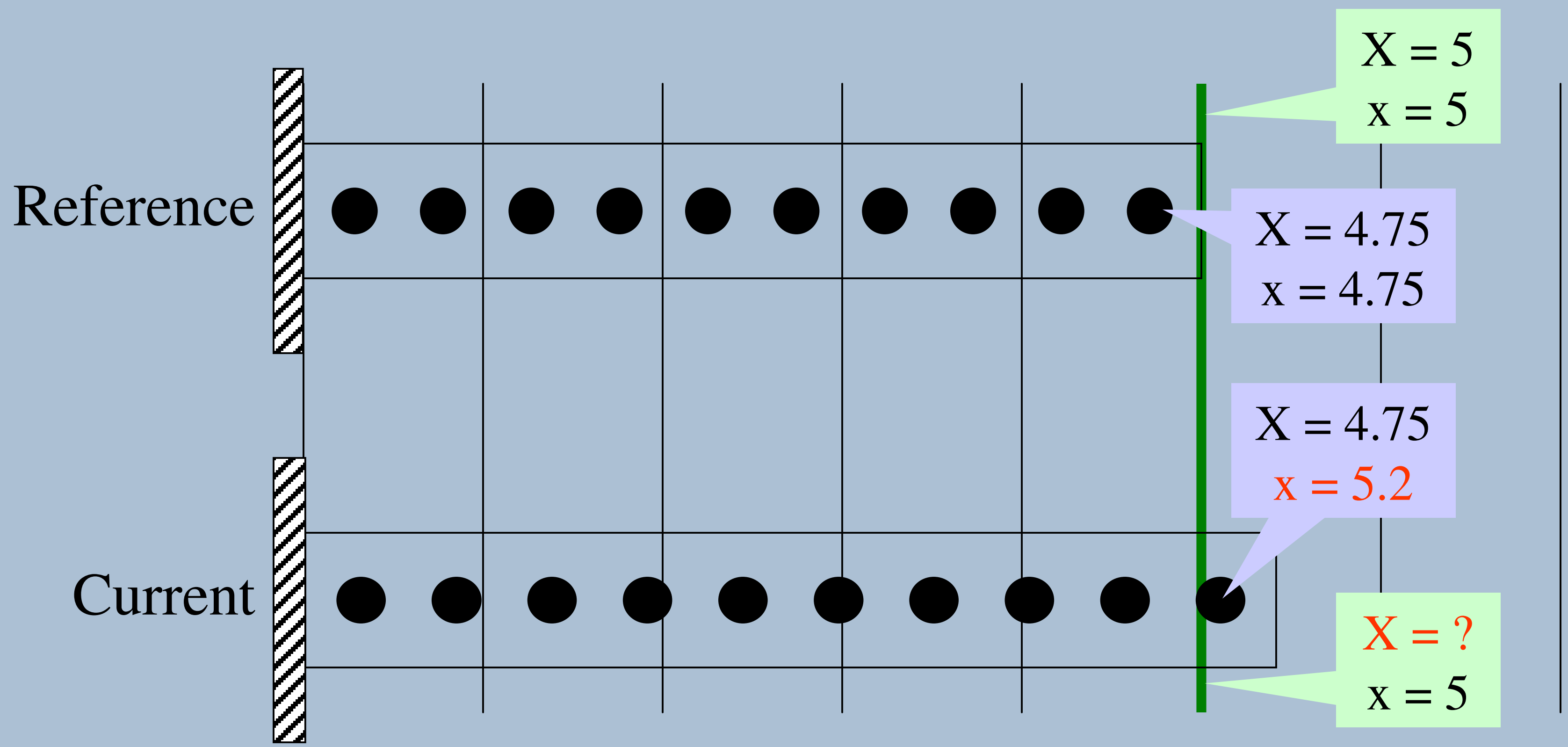
- problem always has sufficient particles per cell
- displacement field is well-behaved (for us A(t) > -1/2)

$$u = \frac{X(2L - X)}{L^2} 0.2 \cos\left(\sqrt{\frac{E}{\rho_0}} \pi t \right)$$

A detour and a review: reference versus current configuration

Particles stationary in reference configuration

Grid stationary in current configuration



Why manufacture solutions in the reference configuration?

– Because boundaries move in the current configuration.

How find the current length and apply boundary?

$$u(x) = \frac{x(2L_0 - x)}{{L_0}^2} A(t)$$

$$\Delta L = u(L_0 + \Delta L) = \frac{(L_0 + \Delta L)(2L_0 - (L_0 + \Delta L))}{{L_0}^2} A(t)$$

This is icky. We can avoid recursive / implicit definitions like the above by using the reference configuration.

Reference Configuration vs Current Configuration

	Reference Configuration "Total Lagrange"	Current Configuration "Updated Lagrange"
Momentum	$\nabla \cdot \mathbf{P} + \rho_0 \mathbf{b} = \rho_0 \mathbf{a}$	$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a}$
Deformation Gradient	$\mathbf{F}(\mathbf{X}) = \mathbf{I} + \nabla \mathbf{u}(\mathbf{X})$	$\mathbf{F}(\mathbf{x}) = \left[\mathbf{I} - \nabla \mathbf{u}(\mathbf{x})\right]^{-1}$
Neo-Hookean	$\mathbf{P} = \lambda \ln J \mathbf{F}^{-1} + \mu \mathbf{F}^{-1}\left(\mathbf{F}\mathbf{F}^{\mathrm{T}} - \mathbf{I}\right)$	$\boldsymbol{\sigma} = \frac{\lambda}{J} \ln J \mathbf{I} + \frac{\mu}{J}\left(\mathbf{F}\mathbf{F}^{\mathrm{T}} - \mathbf{I}\right)$
1D, Poisson = 0	$\mathbf{P} = \frac{E}{2}\left(\mathbf{F}(\mathbf{X}) - \frac{1}{\mathbf{F}(\mathbf{X})}\right)$	$\boldsymbol{\sigma} = \frac{E}{2}\left(\mathbf{F}(\mathbf{x}) - \frac{1}{\mathbf{F}(\mathbf{x})}\right)$

Stress Transformation: $\mathbf{P} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma}$

Return to the 1D Bar: Take Derivatives

Given

Displacement $$u = \frac{X(2L - X)}{L^2} A(t)$$

Deformation Gradient $$F = 1 + \frac{2(L - X)}{L^2} A$$

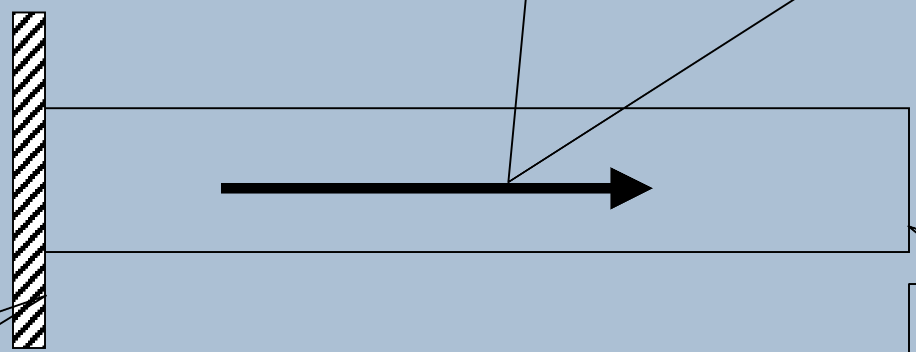
Divergence of Stress $$\nabla \cdot P = -\frac{E}{L^2}\left(1 + \left[1 + \frac{2(L - X)}{L^2} A\right]^{-2}\right) A$$

Solve for b(X) $$b = \frac{1}{L^2}\left[X(2L - X)\ddot{A} - \frac{E}{\rho_0}\left(1 + \left[1 + \frac{2(L - X)}{L^2} A\right]^{-2}\right) A\right]$$

1D Bar: Restate the Problem

Body Force

$$b = \frac{1}{L^2}\left[X(2L-X)\ddot{A} - \frac{E}{\rho_0}\left(1+\left[1+\frac{2(L-X)}{L^2}A\right]^{-2}\right)A \right]$$



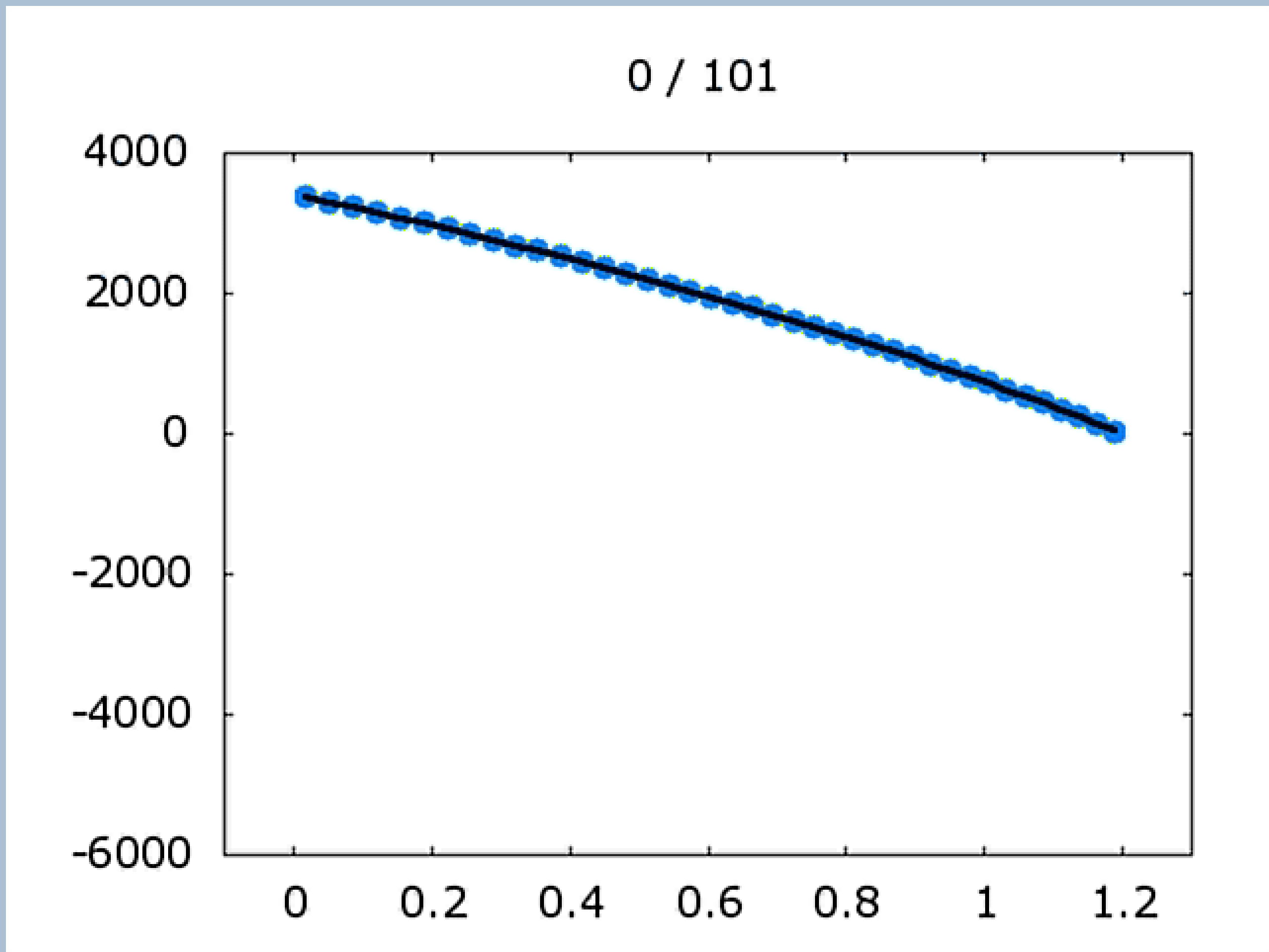
Zero Stress
$\sigma(L,t) = 0$

Zero
Displacement
$u(0,t) = 0$

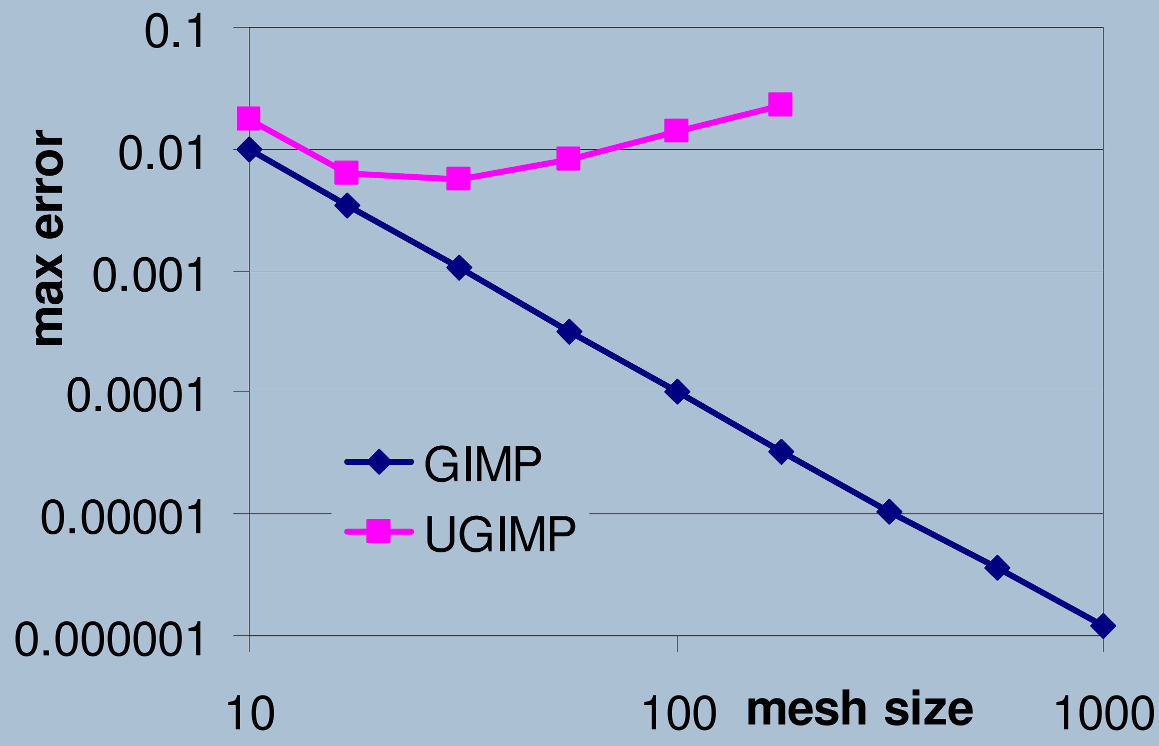
The answer is:

$$u(X) = \frac{X(2L-X)}{L^2}A(t)$$

Solve with GIMP where $A(t) = 0.2\cos\left(\sqrt{\frac{E}{\rho_0}}\pi\, t\right)$



Now we can measure convergence under large deformation – the kind of problem MPM/GIMP is designed to solve



Skeptical Questions

1st Piola-Kirchoff is neither objective nor fully Lagrangian – doesn't that cause problems?

MPM is a first-order, fully non-linear method. It can't be expected to agree with your manufactured solution due to its non-linearity.