# Applying Constraints to the Electrocardiographic Inverse Problem

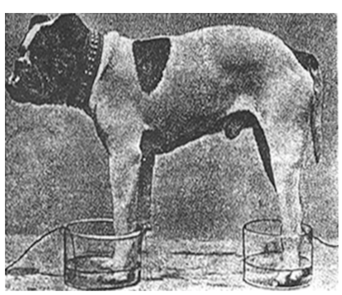
**Rob MacLeod, Dana Brooks*, Yesim Serinagaoglu*, Bulent Yilmaz**

•Cardiovascular Research and Training Institute (CVRTI)

•Northeastern University, CDSP, ECE Department

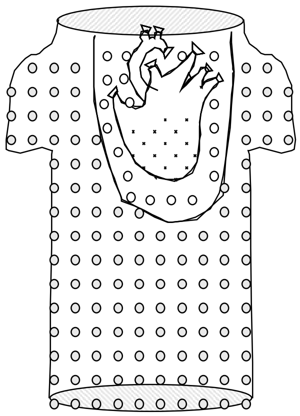
CVRTI/NEU

# Electrocardiography



CVRTI/NEU

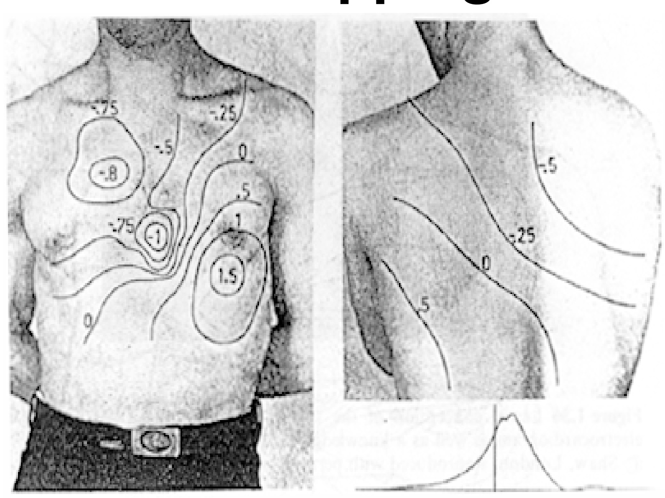
# Electrocardiographic Mapping



- Bioelectric Potentials
- Goals
  - Higher spatial density
  - Imaging modality
- Measurements
  - Body surface
  - Heart surfaces
  - Heart volume

CVRTI/NEU

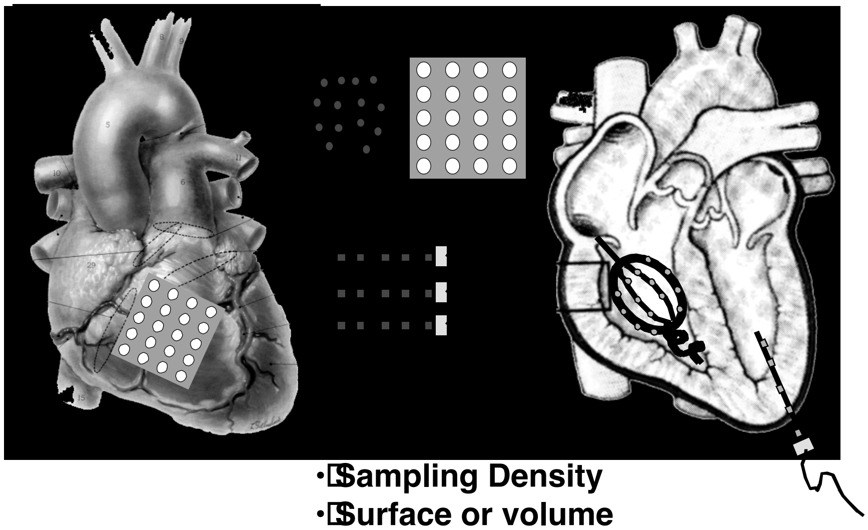
# Body Surface Potential Mapping



Taccardi et al, Circ., 1963

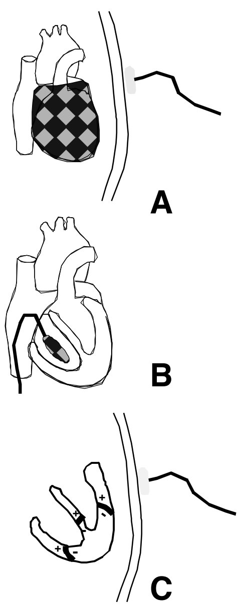
CVRTI/NEU

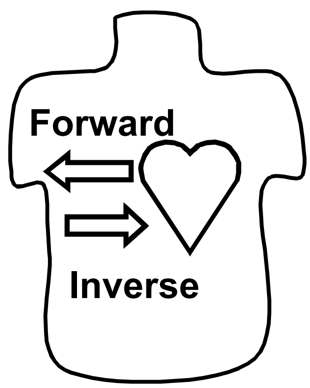
# Cardiac Mapping



- Sampling Density
- Surface or volume

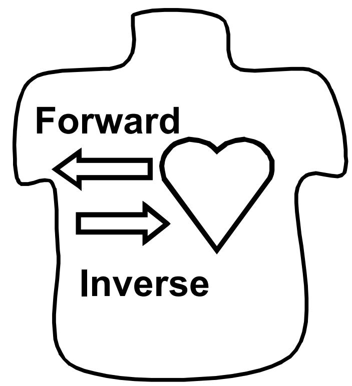
# Inverse Problems in Electrocardiography



# Epicardial Inverse Problem

- **Definition**
  - Estimate sources from remote measurements
- **Motivation**
  - Noninvasive detection of abnormalities
  - Spatial smoothing and attenuation

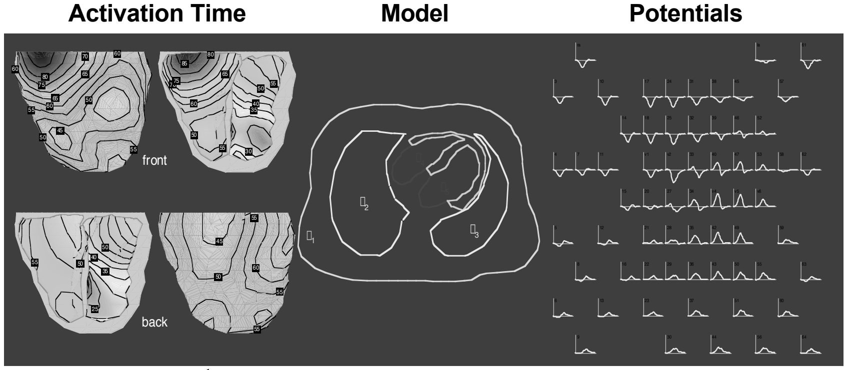



CVRTI/NEU

# Forward/Inverse Problem

**Forward problem**

**Epicardial/Endocardial Activation Time** | **Geometric Model** | **Body Surface Potentials**

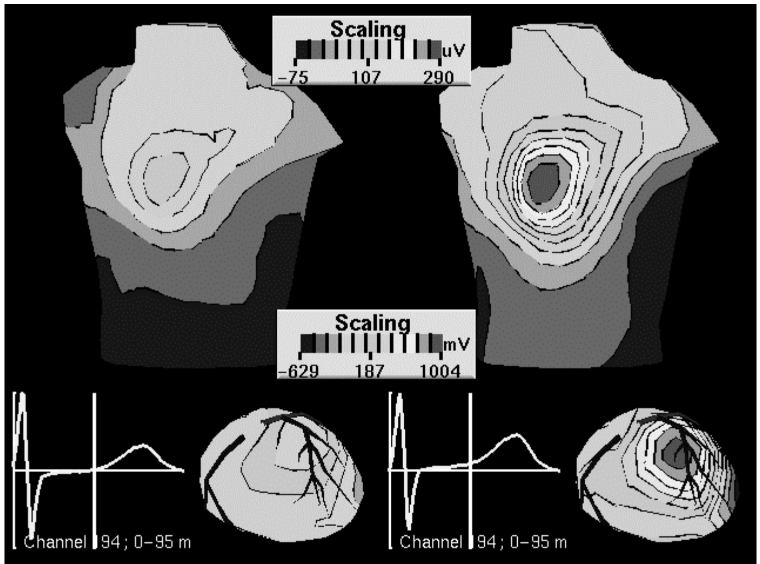



**Inverse problem**

Thom Oostendorp, Univ. of Nijmegen

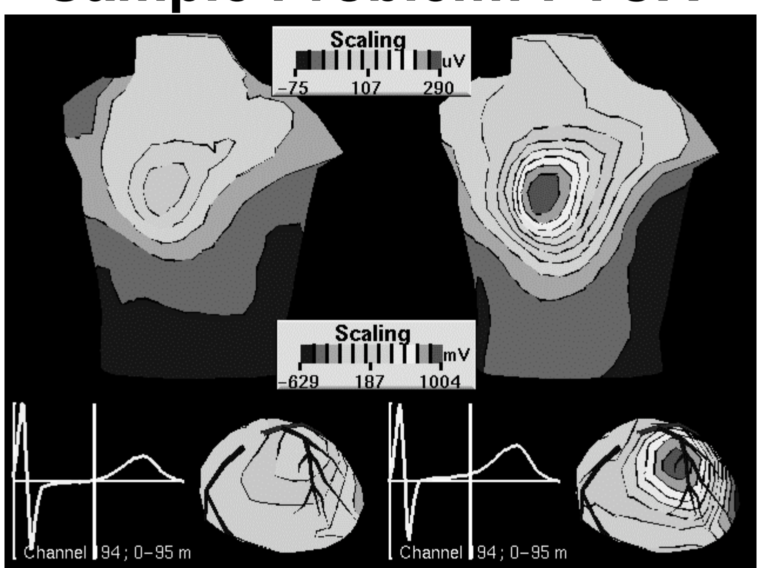
CVRTI/NEU

# Sample Problem: PTCA




CVRTI/NEU


# Sample Problem: PTCA



CVRTI/NEU

## Elements of the Inverse Problem

- **Components**
  - Source description
  - Geometry/conductivity
  - Forward solution
  - “Inversion” method (regularization)
- **Challenges**
  - Inverse is ill-posed
  - Solution ill-conditioned

CVRTI/NEU

## Inverse Problem Research

- **Role of geometry/conductivity**
- **Numerical methods**
- **Improving accuracy to clinical levels**
- **Regularization**
  - *A priori* constraints versus fidelity to measurements

CVRTI/NEU

# Regularization

- **Current questions**
  - **Choice of constraints/weights**
  - **Effects of errors**
  - **Reliability**
- **Contemporary approaches**
  - **Multiple Constraints**
  - **Time Varying Constraints**
  - **Tuned constraints**
  - **Multisource constraints**

CVRTI/NEU

# Tikhonov Approach

Problem formulation

$$\mathbf{y}(k) \quad = \quad \mathbf{A} \cdot \mathbf{h}(k) + \mathbf{e}(k) \qquad\qquad k = 1, 2, \ldots, L$$

Constraint

$$\hat{\mathbf{h}}_\lambda = \arg\min_{\mathbf{x}} \left( \|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2 + \lambda^2 \|\mathbf{R}\mathbf{x}\|^2 \right),$$

Solution

$$\hat{\mathbf{h}}_\lambda = (\mathbf{A}^T\mathbf{A} + \lambda^2\mathbf{R}^T\mathbf{R})^{-1}\mathbf{A}^T\mathbf{y}$$

CVRTI/NEU

# Multiple Constraints

For k constraints

$$\hat{\mathbf{h}}_\lambda = \arg\min_{\mathbf{x}} \left( \|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2 + \sum_{i=1}^{k} \lambda_i^2 \|\mathbf{R}_i \mathbf{x}\|^2 \right)$$

with solution

$$\hat{\mathbf{h}}_\lambda = [\mathbf{A}^T\mathbf{A} + \sum_{i=1}^{k} \lambda_i^2 \mathbf{R}_i^T \mathbf{R}_i]^{-1} \mathbf{A}^T \mathbf{y}.$$

CVRTI/NEU

# Dual Spatial Constraints

For two spatial constraints:

$$\hat{\mathbf{h}}_\lambda = (\mathbf{A}^T\mathbf{A} + \lambda_1^2 \mathbf{R}_1^T \mathbf{R}_1 + \lambda_2^2 \mathbf{R}_2^T \mathbf{R}_2)^{-1} \mathbf{A}^T \mathbf{y}.$$

Note: two regularization factors required

CVRTI/NEU

# Joint Time-Space Constraints

Redefine y, h, A:

$$\overline{\mathbf{y}} = \overline{\mathbf{A}\mathbf{h}} + \overline{\mathbf{e}}$$

$$\overline{\mathbf{A}} = \begin{pmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A} & \cdots & \mathbf{0} \\ & & & \cdots & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A} \end{pmatrix}.$$

And write a new minimization equation:

$$\widehat{\mathbf{h}} = \arg\min_{\overline{\mathbf{x}}} \left( \|\overline{\mathbf{A}}\overline{\mathbf{x}} - \overline{\mathbf{y}}\|^2 + \sum_{i=1}^{k_s} \lambda_i^2 \|\overline{\mathbf{R}}_i \overline{\mathbf{x}}\|^2 + \sum_{i=1}^{k_t} \eta_i^2 \|\overline{\mathbf{T}}_i \overline{\mathbf{x}}\|^2 \right).$$

CVRTI/NEU

# Joint Time-Space Constraints

General solution:

$$\hat{\overline{\mathbf{h}}} = (\overline{\mathbf{A}}^T\overline{\mathbf{A}} + \sum_{i=1}^{k_s} \lambda_i^2 \overline{\mathbf{R}}_i^T \overline{\mathbf{R}}_i + \sum_{i=1}^{k_t} \eta_i^2 \overline{\mathbf{T}}_i^T \overline{\mathbf{T}}_i)^{-1} \overline{\mathbf{A}}^T \overline{\mathbf{y}}$$

For a single space and time constraint:

$$\begin{aligned} \hat{\overline{\mathbf{h}}} &= (\overline{\mathbf{A}}^T\overline{\mathbf{A}} + \lambda^2 \overline{\mathbf{R}}^T \overline{\mathbf{R}} + \eta^2 \overline{\mathbf{T}}^T \overline{\mathbf{T}})^{-1} \overline{\mathbf{A}}^T \overline{\mathbf{y}} \\ &= [\mathbf{I}_L \otimes (\mathbf{A}^T\mathbf{A}) + \lambda^2 \mathbf{I}_L \otimes \mathbf{R}^T\mathbf{R} + \eta^2 (\mathbf{T}^T\mathbf{T}) \otimes \mathbf{I}_N]^{-1} \cdot \\ &\quad \cdot (\mathbf{I}_L \otimes \mathbf{A}^T) \overline{\mathbf{y}}. \end{aligned}$$

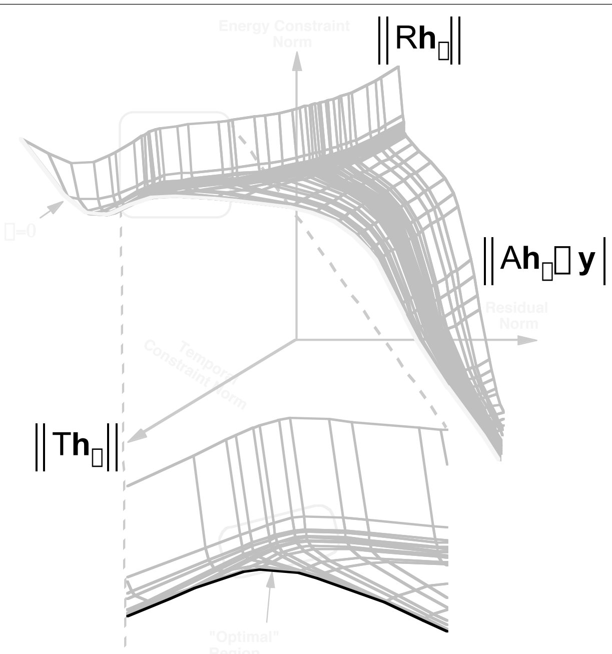
Note: two regularization factors and implicit temporal factor

CVRTI/NEU

# Determining Weights

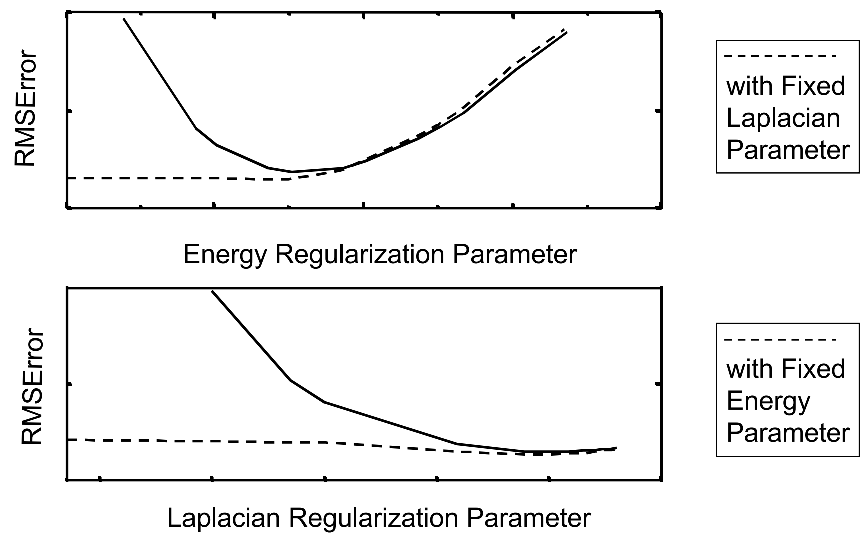
- **Based on *a posteriori* information**
- ***Ad hoc* schemes**
  - **CRESO: composite residual and smooth operator**
  - **BNC: bounded norm constraint**
  - **AIC: Akaike information criterion**
  - **L-curve: residual norm vs. solution seminorm**

# L-Surface



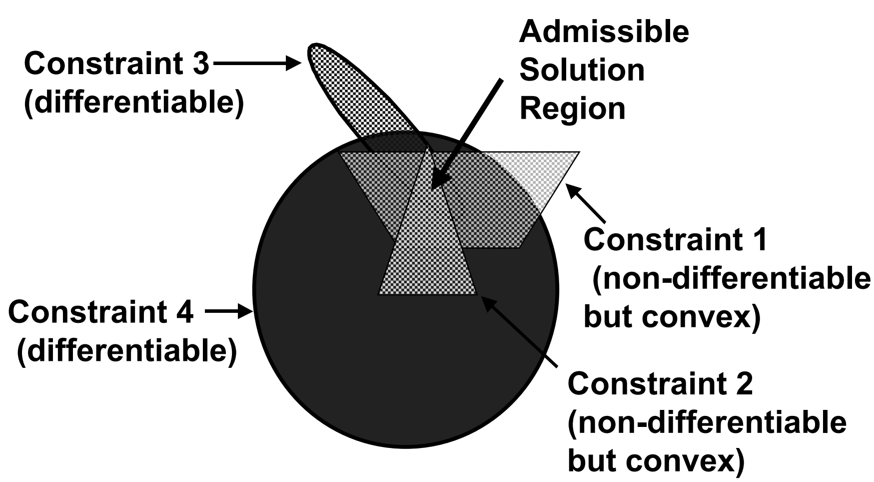

- **Natural extension of single constraint approach**
- **"Knee" point becomes a region**

# Joint Regularization Results




CVRTI/NEU

# Admissible Solution Approach




CVRTI/NEU

# Single Constraint

Define $\phi(\mathbf{x})$ s.t.

$$\phi(\mathbf{x}) : \mathcal{R}^N \rightarrow \mathcal{R}$$

with the constraint such that

$$\phi(\mathbf{x}) - \epsilon < 0.$$

that satisfies the convex condition

$$\phi(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha\phi(\mathbf{x}) + (1-\alpha)\phi(\mathbf{y}) \qquad \forall\alpha \in [0,1].$$

CVRTI/NEU

# Multiple Constraints

Define multiple constraints $\phi_i(\mathbf{x})$

$$(\phi_i(\mathbf{x}) - \epsilon_i) \in \mathcal{H}, \text{ for } i = 1, 2, \ldots, m.$$

so that the set of these

$$\{\mathbf{x} : \phi(\mathbf{x}) < 0\}$$

represents the intersection of all constraints. When they satisfy the joint condition

$$\phi(\mathbf{x}) \leq 0$$

Then the resulting **x** is the *admissible solution*

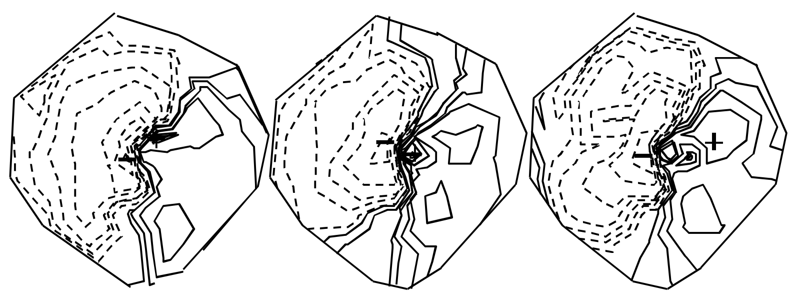
CVRTI/NEU

# Examples of Constraints

- Residual constraint $\phi(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2$
- Regularization constraints $\phi(\mathbf{x}) = \|\mathbf{R}\mathbf{x}\|_2^2$
- Tikhonov constraints $\phi_\lambda(\mathbf{x}) = \left\| \begin{pmatrix} \mathbf{A} \\ \sqrt{\lambda}\mathbf{R} \end{pmatrix} \mathbf{x} - \begin{pmatrix} \mathbf{b} \\ \mathbf{0} \end{pmatrix} \right\|_2^2$
- Spatiotemporal constraints
- Weighted constraints
- Novel constraints

CVRTI/NEU

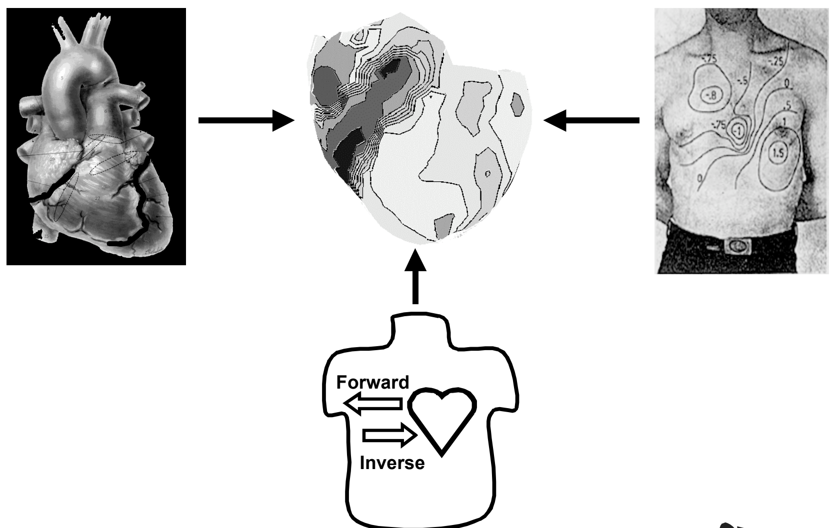
# Admissible Solution Results



Original | Regularized | Admissible Solution

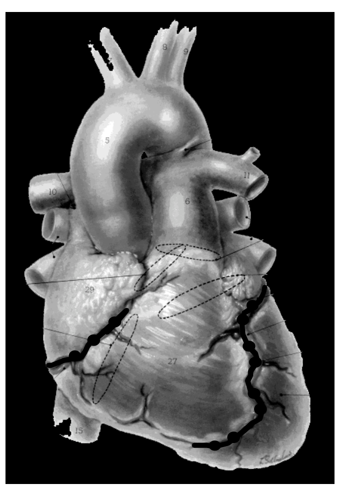
CVRTI/NEU

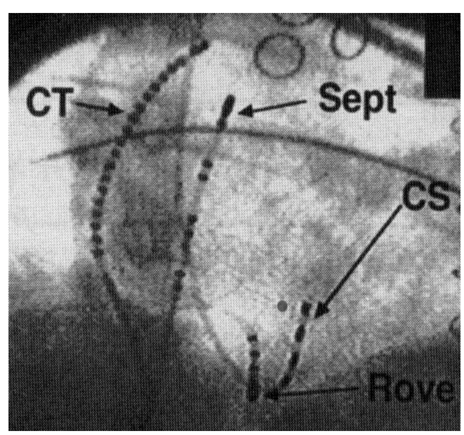
# Combining Information Sources




CVRTI/NEU

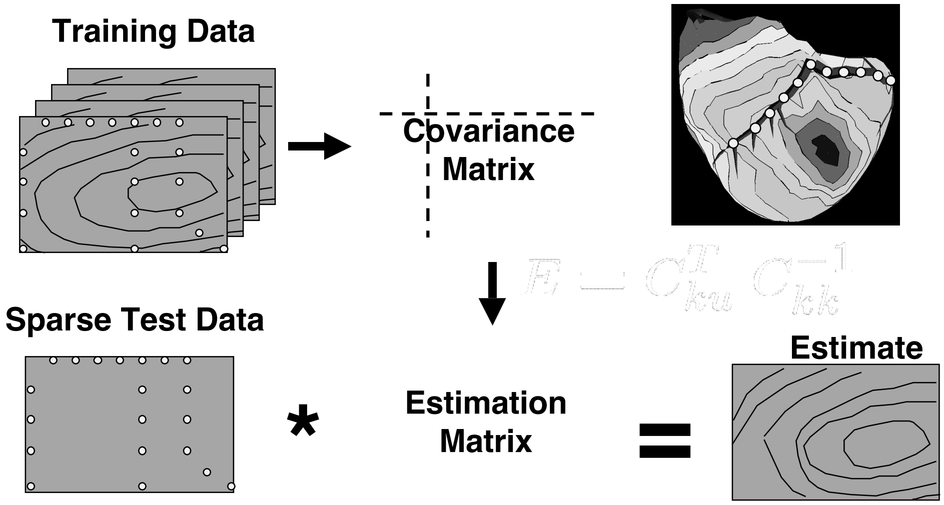
# Venous Catheter Based Mapping






CVRTI/NEU

# Statistical Estimation

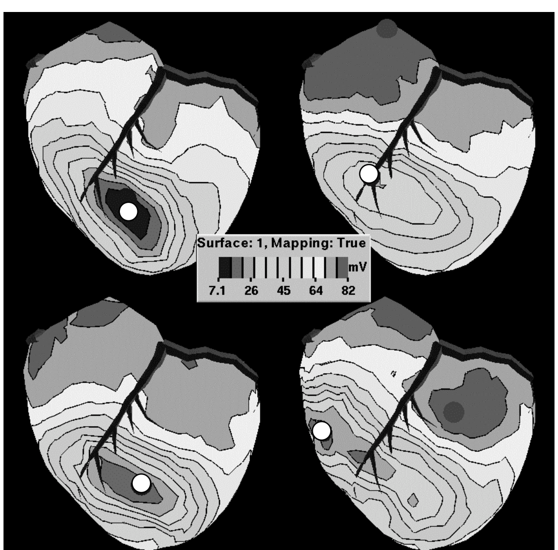



$$F = C_{ku}^T C_{kk}^{-1}$$

$$t_u = F t_k$$

CVRTI/NEU

# Estimated Activation Maps




- Training set composition
- Lead selection

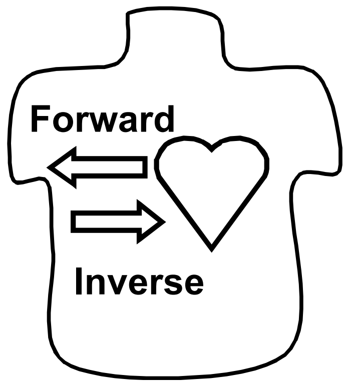
CVRTI/NEU

# Augmented Inverse Problem

**Torso geometry**
**+**
**Body-Surface Potentials**
**+**
**Sparse Epicardial Potentials**
**+**
**Inverse Solution**

---

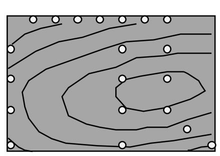
**Epicardial Map**



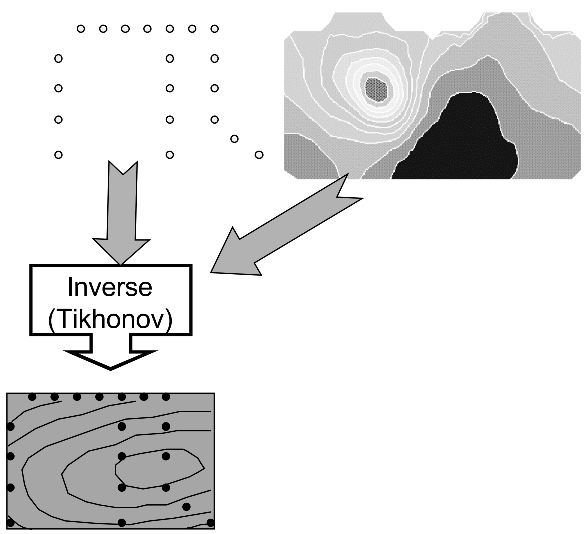

CVRTI/NEU

# Subtraction Approach

***Unknown*** ***Known***

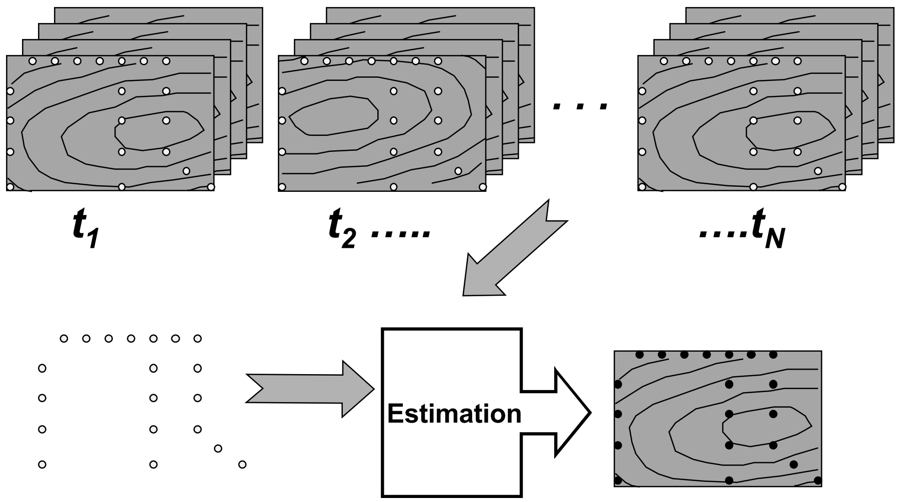


1) Subtract known epicardial potentials
2) Solve reduced inverse problem



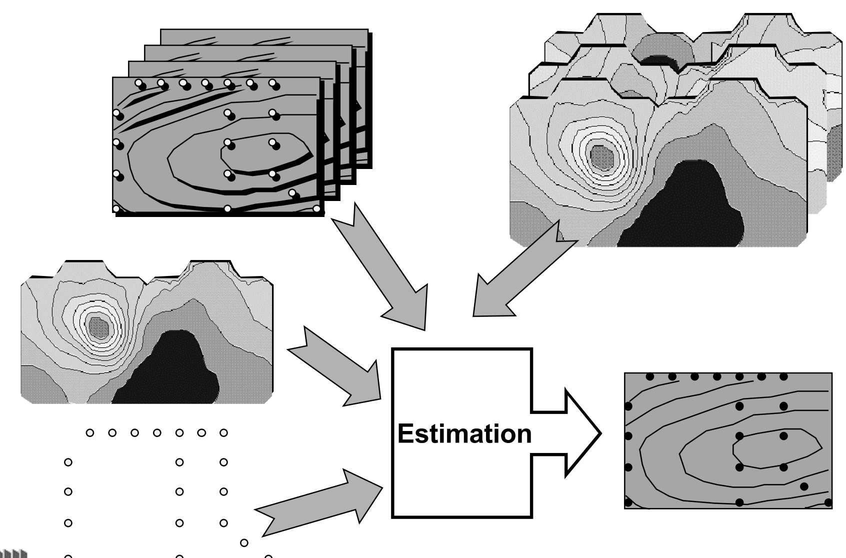

CVRTI/NEU

# Epicardial Estimation



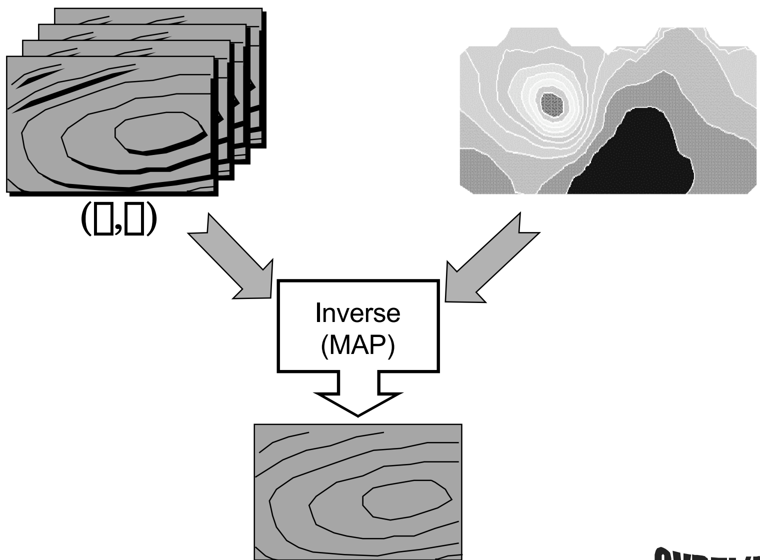

CVRTI/NEU

# Combined Estimation



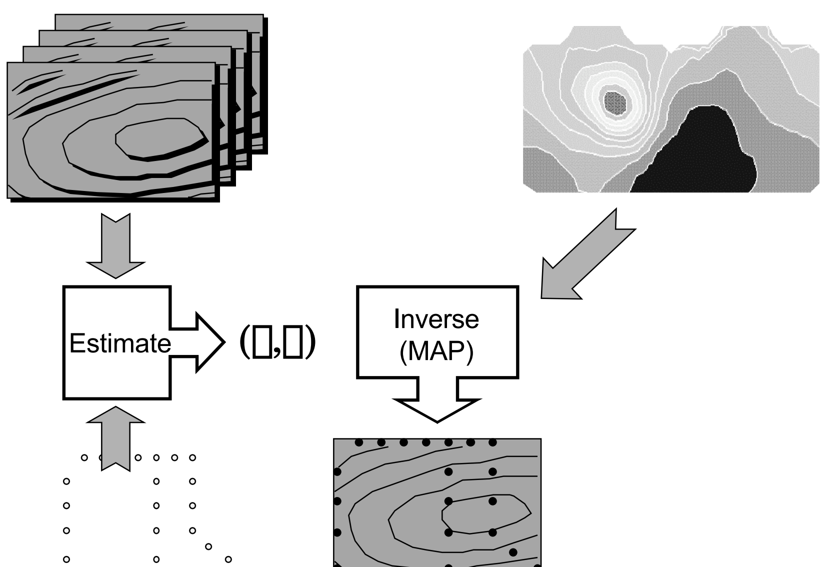

CVRTI/NEU

# Bayesian Approach



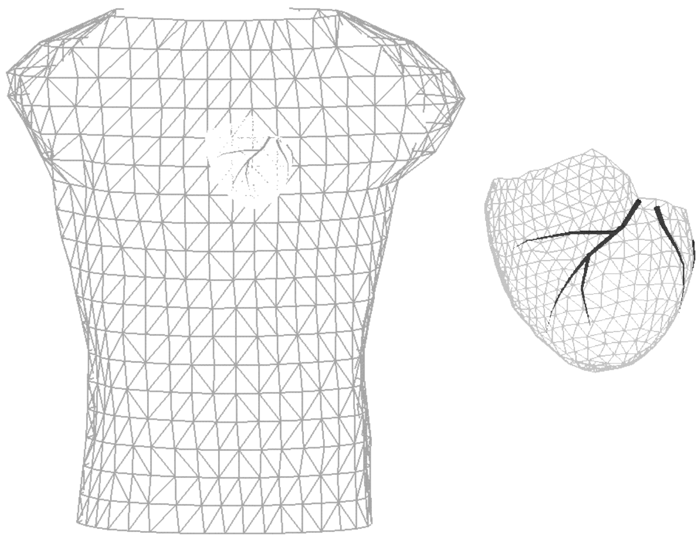

CVRTI/NEU

# Hybrid Approach



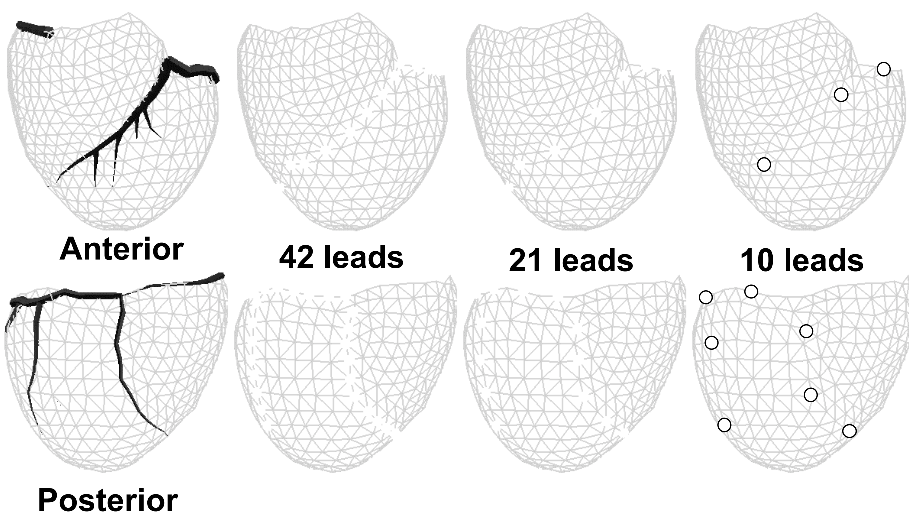

CVRTI/NEU

# Tank/Heart Geometry



CVRTI/NEU

# Test Lead Sets




CVRTI/NEU

# Simulation Study

- **490 lead measured sock data**
- **Surrogate catheter potentials**
  - **42 sites**
  - **+ Gaussian noise**
- **Torso potentials**
  - **Calculated noise-free using forward model**
  - **+ Gaussian noise**

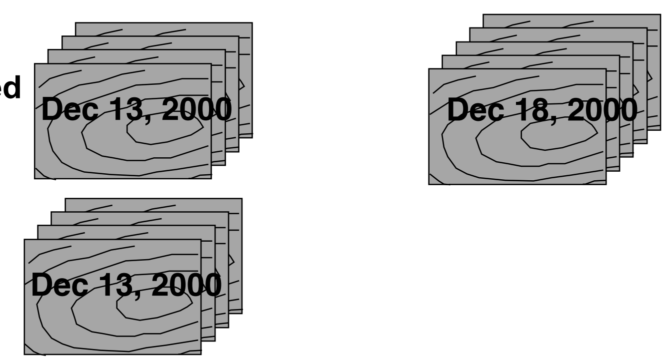
CVRTI/NEU

# Leave-One-Experiment Out Protocol

**Training Data** **Test Data**

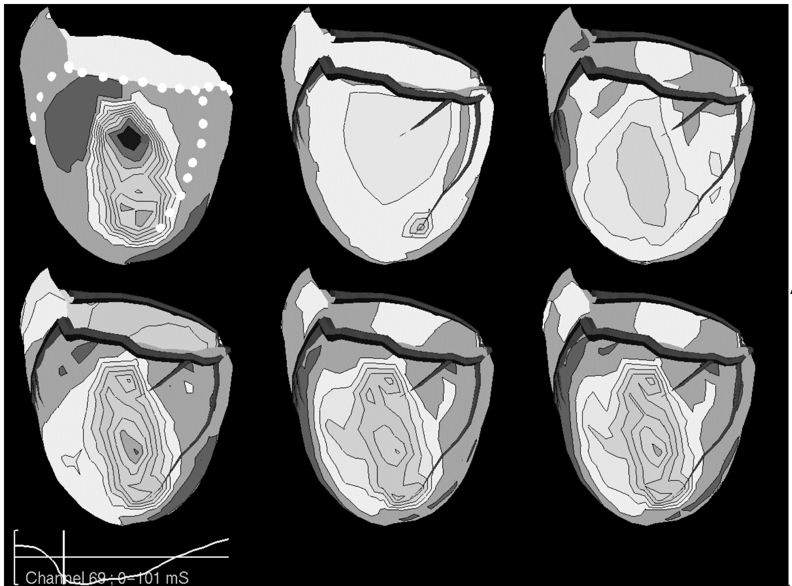
**LV Paced Beats**

**Mixed Paced Beats**




CVRTI/NEU

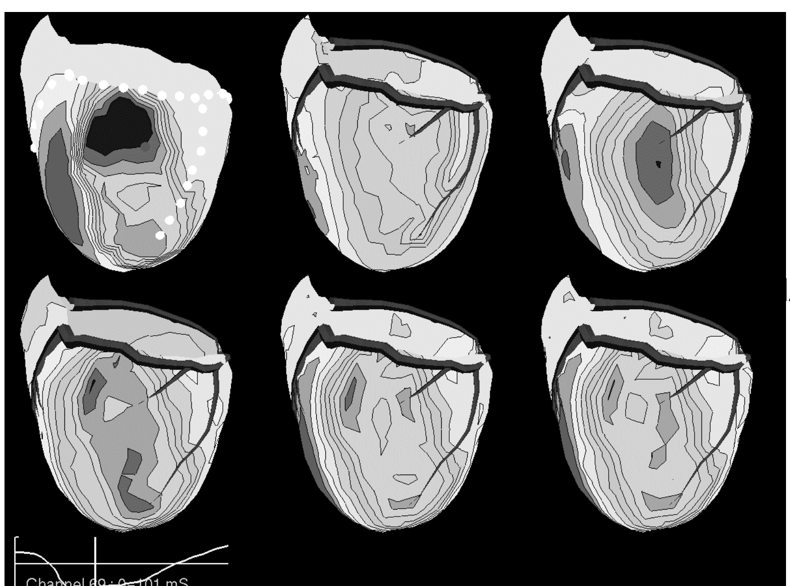
# LV Pacing (LV-23 ms)




AP

CVRTI/NEU

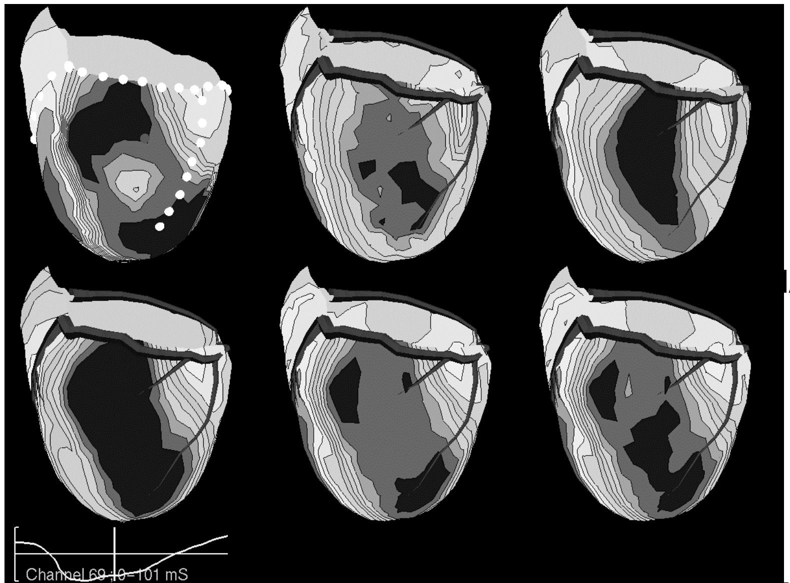
# LV Pacing (LV-38 ms)




AP

CVRTI/NEU

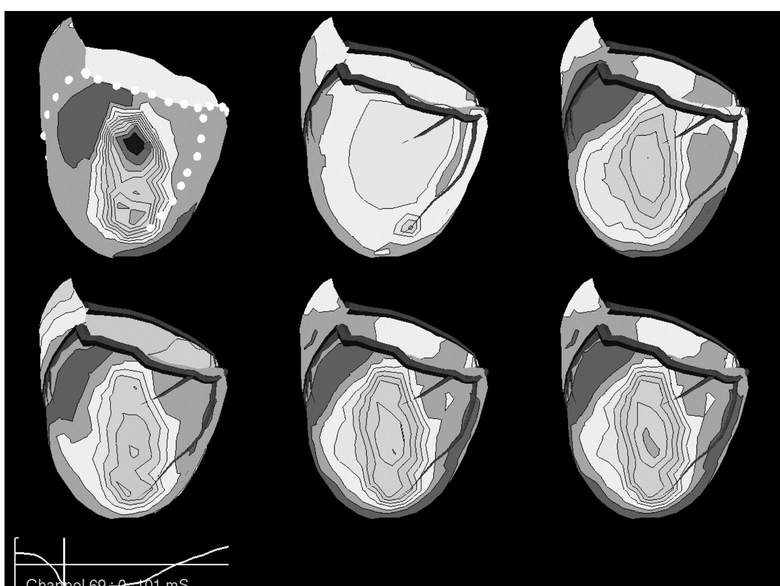
## LV Pacing (LV-47 ms)




AP

CVRTI/NEU

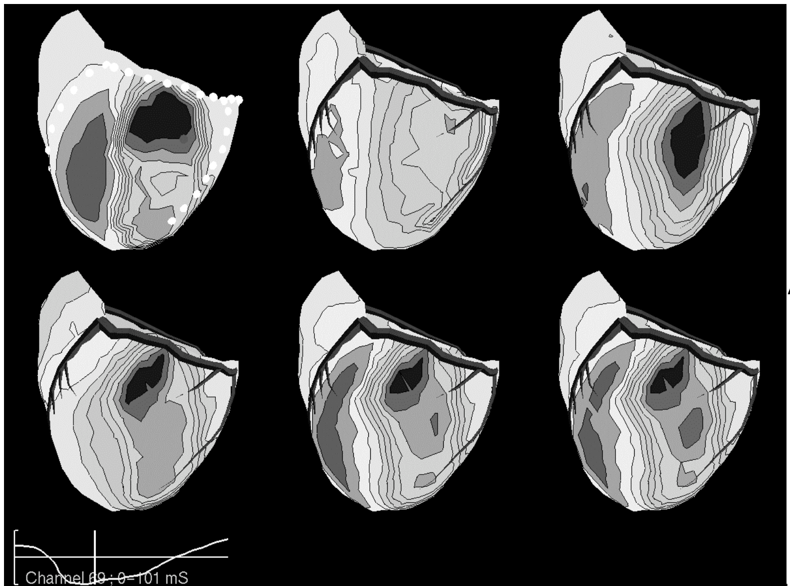
## LV Pacing (Mixed-23 ms)




AP

CVRTI/NEU

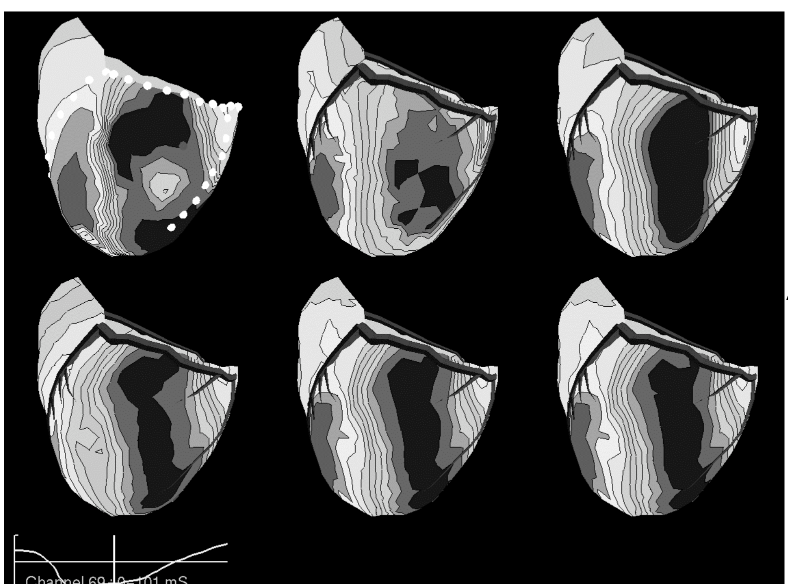
## LV Pacing (Mixed-38 ms)




AP

CVRTI/NEU

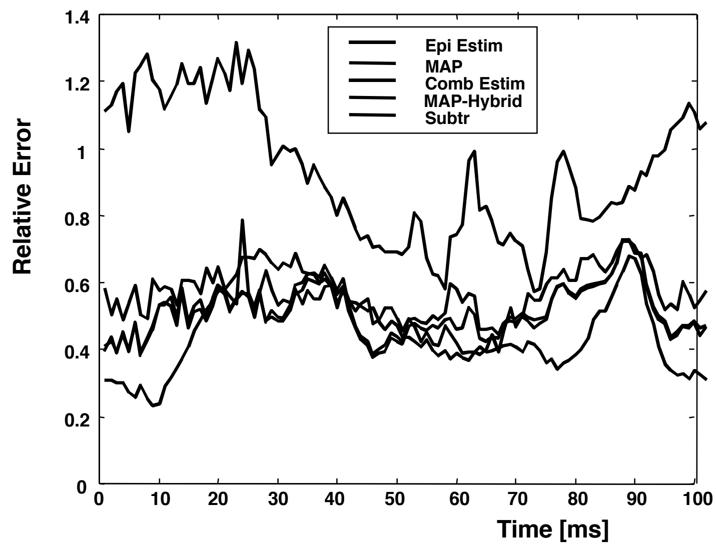
## LV Pacing (Mixed-47 ms)




AP

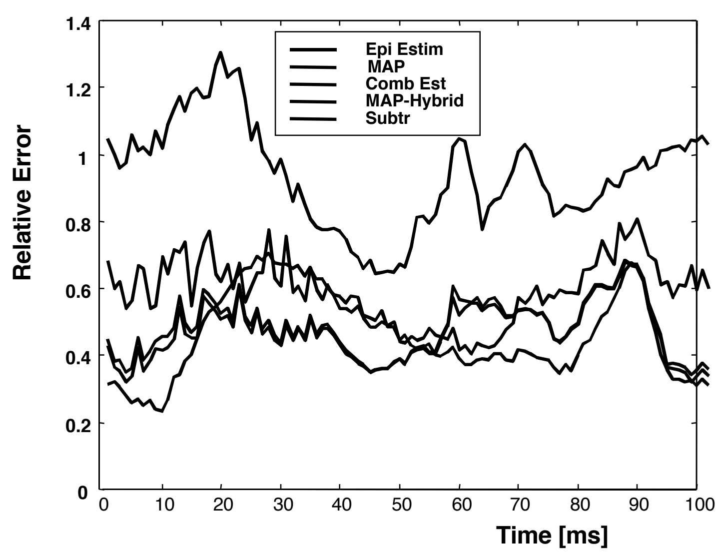
CVRTI/NEU

# Relative Error (31-LV)




CVRTI/NEU

# Relative Error (31-Mixed)




CVRTI/NEU

## Estimation Findings

- Estimation alone: noisy, unstable results
- Estimation + inverse: smoothing improves stability

CVRTI/NEU

## Inverse Solution Findings

- All solutions better than simple Tikhonov
- MAP usually improved with addition of catheter measurements (Hybrid MAP)

CVRTI/NEU

# Role of Statistics (Training)

- Generally helps
- But can add artifacts, e.g., spurious breakthroughs or wavefronts
- Torso potentials can reduce artifacts

CVRTI/NEU


# Acknowledgements

- CVRTI
  - Bruno Taccardi
  - Rich Kuenzler
  - Bob Lux
  - Phil Ershler
  - Yonild Lian
  - Shibaji Shome
  - Lucas Lorenzo
- CDSP
  - Dana Brooks
  - Ghandi Ahmad


www.cvrti.utah.edu/~macleod

www.censsis.neu.edu

CVRTI/NEU