Boundary Element Method (BEM)

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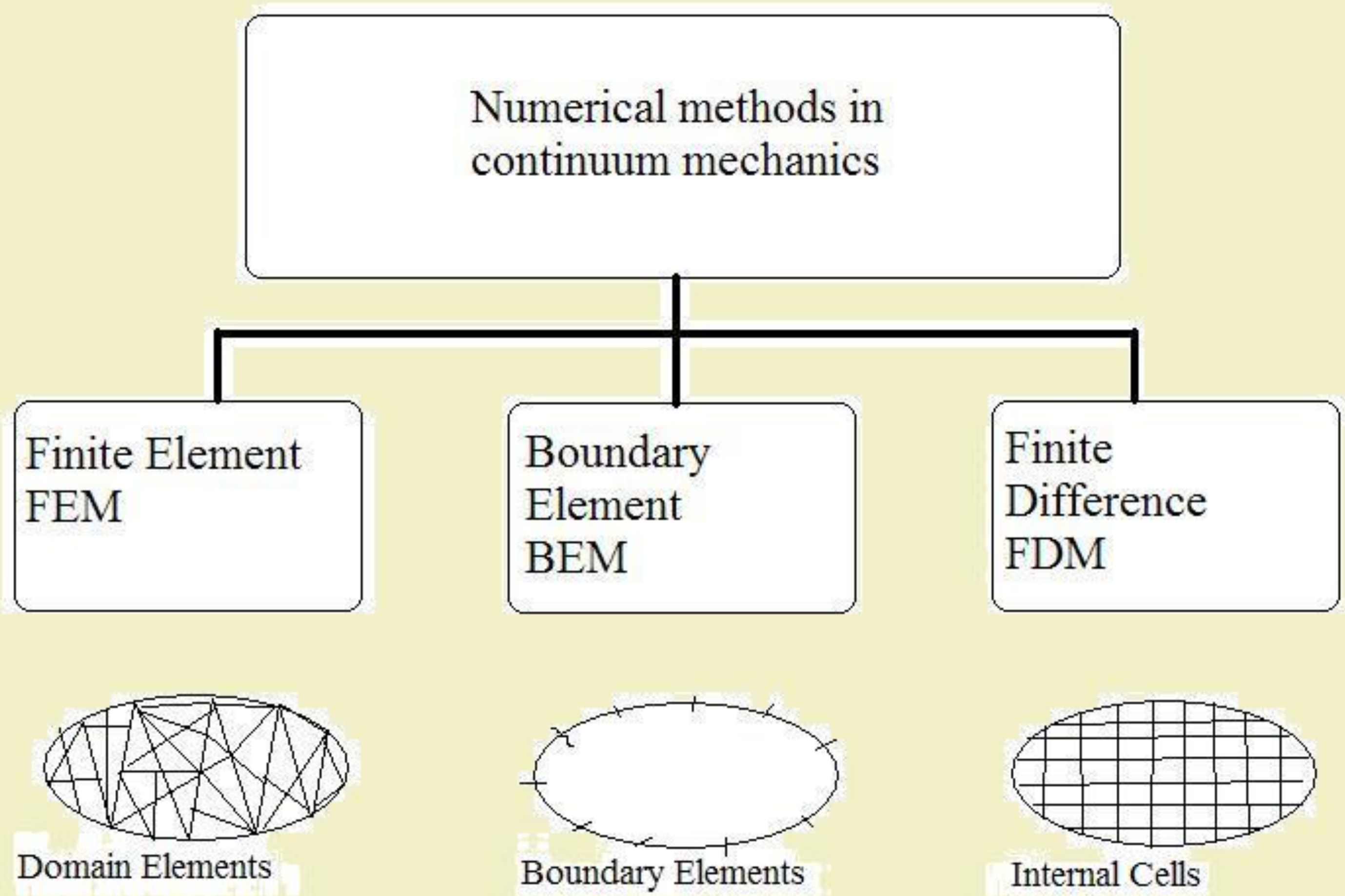
HG 6.96 (TU/e)

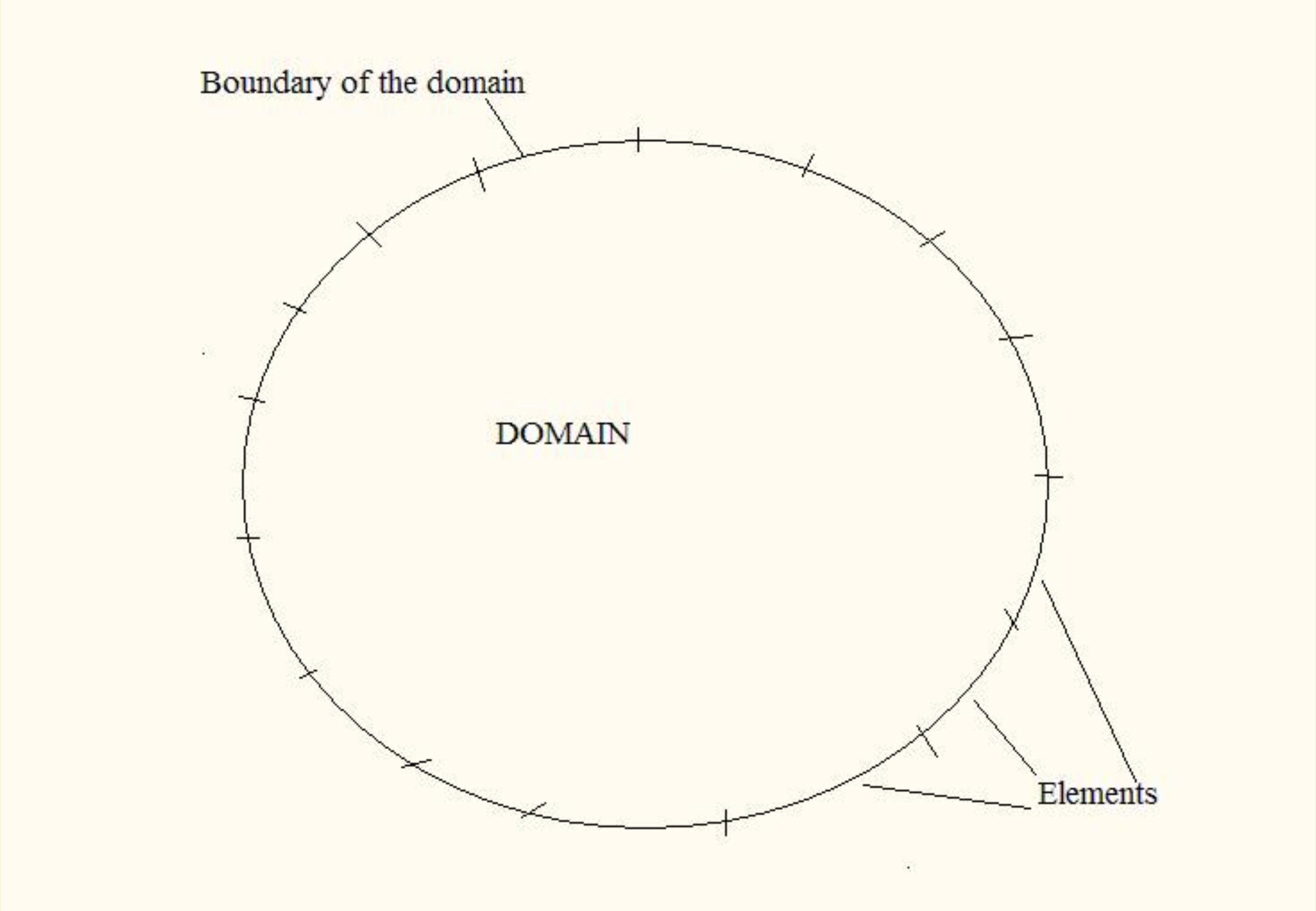
Talk Overview

- The idea of BEM and its advantages
- The 2D potential problem
- Numerical implementation

Idea of BEM


Numerical methods in
continuum mechanics
Finite Element
FEM
Boundary
Element
BEM
Finite
Difference
FDM
Domain Elements
Boundary Elements
Internal Cells

Idea of BEM



Advantages of BEM

1) Reduction of problem dimension by 1

- Less data preparation time.
- Easier to change the applied mesh.
- Useful for problems that require re-meshing.

Advantages of BEM

2) High Accuracy

- Stresses are accurate as there are no approximations imposed on the solution in interior domain points.
- Suitable for modeling problems of rapidly changing stresses.

Advantages of BEM

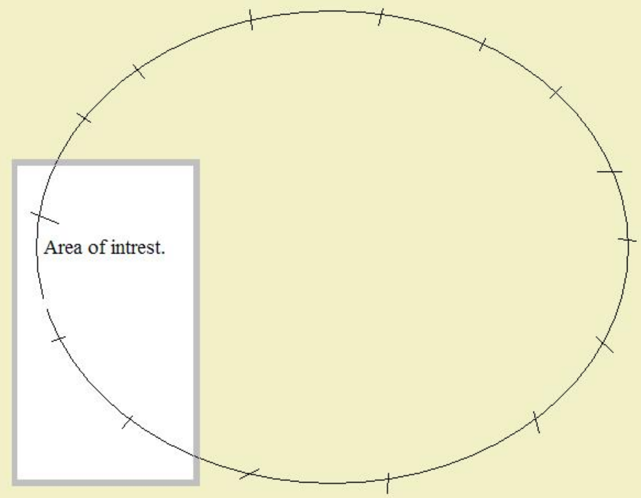
3) Less computer time and storage

- For the same level of accuracy as other methods BEM uses less number of nodes and elements.

Advantages of BEM

4) Filter out unwanted information.

- Internal points of the domain are optional.
- Focus on particular internal region.



- Further reduces computer time.

Advantages of BEM

1. Reduction of problem dimension by 1.
2. High Accuracy.
3. Less computer time and storage.
4. Filter out unwanted information and so focus on section of the domain you are interested in.

BEM is an attractive option.

The 2D potential problem

- Where can BEM be applied?
- Two important functions.
- Description of the domain.
- Mapping of higher to lower dimensions.
- Satisfaction of the Laplace equations and how to deal with a singularity.
- The boundary integral equation (BIE)

The 2D potential problem

Where can BEM be applied?

Where any potential problem is governed by a differential equation that satisfies the Laplace equation. (or any other behavior that has a related fundamental solution)

e.g. The following can be analyzed with the Laplace equation: fluid flow, torsion of bars, diffusion and steady state heat conduction.

The 2D potential problem

The Laplace equation for 2D

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$\nabla^2 = \nabla.\nabla =$ Laplacian operator

$\phi =$ Potential function

$x, y =$ Cartesian coordinate axis

The 2D potential problem

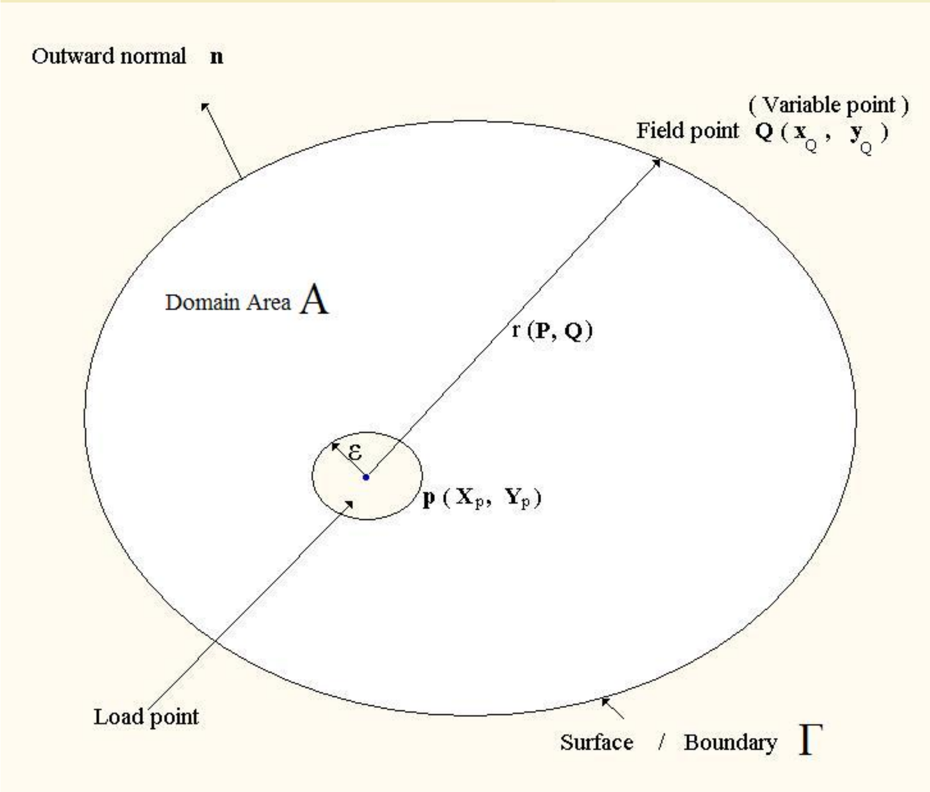
Two important functions.

$\phi =$ The function describing the property under analysis. e.g. heat. (Unknown)

$\nabla^2 \lambda = \partial(Q - P)$ The fundamental solution of the Laplace equation. (These are well known)

The 2D potential problem

Description of the domain



Fundamental solution of the 2D Laplace equation for a concentrated source point at p is

$$\lambda(p,Q) = \frac{1}{2\pi}\ln\left[\frac{1}{r(p,Q)}\right]$$

Where

$$r(p,Q) = \sqrt{\left(\left(X_p - x_Q\right)^2 + \left(Y_p - y_Q\right)^2\right)}$$

The 2D potential problem

Mapping of higher to lower dimensions

- Boundary of any domain is of a dimension 1 less than of the domain.
- In BEM the problem is moved from within the domain to its boundary.
- This means you must, in this case, map Area to Line.
- The well known ‘Greens Second Identity’ is used to do this.

$$\int_A \left(\phi\nabla^2\lambda - \lambda\nabla^2\phi\right)dA = \int_\Gamma \left(\phi\frac{\partial\lambda}{\partial n} - \lambda\frac{\partial\phi}{\partial n}\right)d\Gamma$$

ϕ, λ have continuous 1st and 2nd derivatives.

ϕ unknown potential at any point.

λ known fundamental solution at any point.

n unit outward normal. $\frac{\partial}{\partial n}$ derivative in the direction of normal.

The 2D potential problem

Satisfying the Laplace equation

The unknown ϕ will satisfy $\nabla^2\phi = 0$ everywhere in the solution domain.

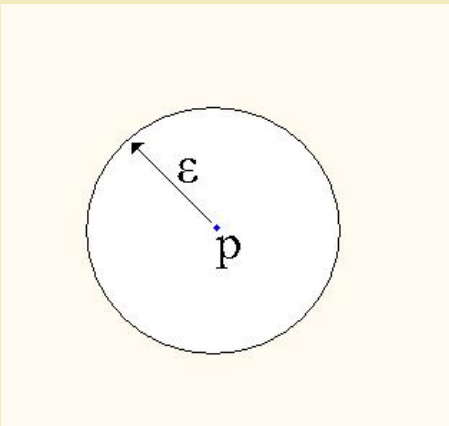
The known fundamental solution λ satisfies $\nabla^2\lambda = 0$ everywhere except the point p where it is singular.

$$\lambda(p,Q) = \frac{1}{2\pi}\ln\left[\frac{1}{r(p,Q)}\right]$$

$$r(p,Q) = \sqrt{\left(\left(X_p - x_Q\right)^2 + \left(Y_p - y_Q\right)^2\right)}$$

The 2D potential problem

How to deal with the singularity



- Surround p with a small circle of radius ε , then examine solution as ε → 0
- New area is $(A - A_\varepsilon)$
- New boundary is $(\Gamma + \Gamma_\varepsilon)$

$$\int_{A-A\varepsilon}\left(\phi\nabla^2\lambda - \lambda\nabla^2\phi\right)dA = \int_{\Gamma+\Gamma\varepsilon}\left(\phi\frac{\partial\lambda}{\partial n} - \lambda\frac{\partial\phi}{\partial n}\right)d\Gamma$$

Within area $(A - A\varepsilon)$ $\nabla^2\phi = 0$ & $\nabla^2\lambda = 0$

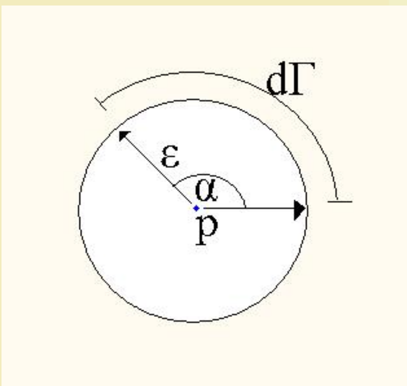
The left hand side of the equation is now 0 and the right is now …

The 2D potential problem

How to deal with the singularity

$$0 = \int_{\Gamma} \left(\phi \frac{\partial \lambda}{\partial n} - \lambda \frac{\partial \phi}{\partial n} \right) d\Gamma + \int_{\Gamma \varepsilon} \left(\phi \frac{\partial \lambda}{\partial n} - \lambda \frac{\partial \phi}{\partial n} \right) d\Gamma$$

The second term must be evaluated and to do this let $d\Gamma = \varepsilon d\alpha$



And use the fact that $\dfrac{\partial \lambda}{\partial n} = \dfrac{\partial \lambda}{\partial r} \cdot \dfrac{\partial r}{\partial n} = \dfrac{1}{2\pi r}$

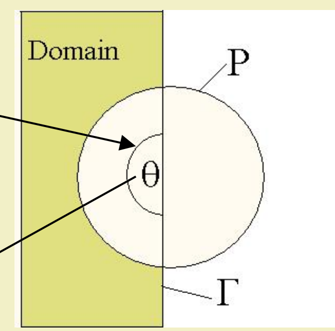
The 2D potential problem

How to deal with the singularity

$$\int_{\Gamma\varepsilon}\left(\phi\frac{\partial\lambda}{\partial n}-\lambda\frac{\partial\phi}{\partial n}\right)d\Gamma=\frac{1}{2\pi}\int_0^{2\pi}\left[\phi\left(\frac{1}{\varepsilon}\right)-\ln\left(\frac{1}{\varepsilon}\right)\frac{\partial\phi}{\partial n}\right]\varepsilon d\alpha$$

$$=\frac{1}{2\pi}(2\pi\phi)=1.\phi$$

$$C(P)=\begin{cases}1 & \text{Evaluated with p in the domain,}\\ 1/2 & \text{on the boundary (Smooth surface),}\\ 0 & \text{and outside the boundary.}\end{cases}$$



$$C(P)=\frac{\theta}{2\pi}$$ For coarse surfaces

The 2D potential problem

The boundary integral equation

$$C(P)\phi(P) = \int_\Gamma K_2(P,Q)\frac{\partial \phi}{\partial n}\, d\Gamma(Q) - \int_\Gamma K_1(P,Q)\phi(Q)\, d\Gamma(Q)$$

Where K1 and K2 are the known fundamental solutions and are equal to

$$K_1(P,Q) = \frac{\partial \lambda(P,Q)}{\partial n}$$

$$K_2(P,Q) = \lambda(P,Q)$$

$$C(P) = \frac{\theta}{2\pi}$$

The 2D potential problem

- BEM can be applied where any potential problem is governed by a differential equation that satisfies the Laplace equation.

  In this case the 2D form.

- A potential problem can be mapped from higher to lower dimension using Green's second identity.

- Shown how to deal with the case of the singularity point.

- Derived the boundary integral equation (BIE)

$$C(P)\phi(P) = \int_\Gamma K_2(P,Q)\frac{\partial \phi}{\partial n}\, d\Gamma(Q) - \int_\Gamma K_1(P,Q)\phi(Q)\, d\Gamma(Q)$$

Numerical Implementation

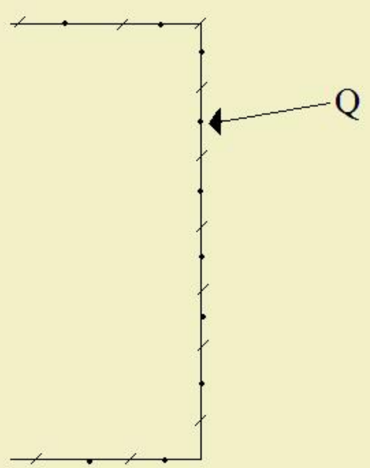
- Dirichlet, Neumann and mixed case.
- Discretisation
- Reduction to a form Ax=B

Numerical Implementation

Dirichlet, Neumann and mixed case.

$$C(P)\phi(P) = \int_\Gamma K_2(P,Q)\frac{\partial\phi}{\partial n}\,d\Gamma(Q) - \int_\Gamma K_1(P,Q)\phi(Q)\,d\Gamma(Q)$$

The unknowns of the above are values on the boundary and are $\phi, \dfrac{\partial\phi}{\partial n}$



Dirichlet Problem

ϕ is given every point Q on the boundary.

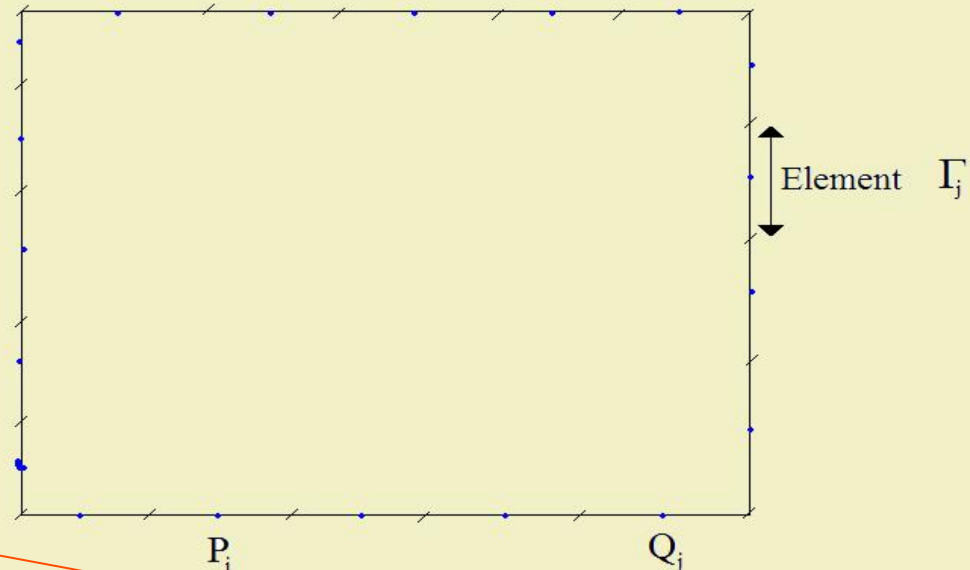
Neumann Problem

$\dfrac{\partial\phi}{\partial n}$ is given every point Q on the boundary.

Mixed case – Either are given at point Q

Numerical Implementation

Discretisation



Unknowns

$$\tfrac{1}{2}\phi(P_i) = \sum_{j=1}^{N} \frac{\partial \phi(Q_j)}{\partial n} \int_{\Gamma_j} K_2(P_i,Q_j)\,d\Gamma_j - \sum_{j=1}^{N} \phi(Q_j) \int_{\Gamma_j} K_1(P_i,Q_j)\,d\Gamma_j$$

Numerical Implementation

Discretisation

Let $K_{1ij} = \int_{\Gamma_j} K_1(P_i, Q_j)\,d\Gamma_j$ $\quad K_{2ij} = \int_{\Gamma_j} K_2(P_i, Q_j)\,d\Gamma_j$

Unknowns

$$\tfrac{1}{2}\phi(P_i) = \sum_{j=1}^{N} \frac{\partial \phi(Q_j)}{\partial n} K_{2ij} - \sum_{j=1}^{N} \phi(Q_j) K_{1ij}$$

Numerical Implementation

$$\phi(P_i) = \phi(Q_j) \quad when \quad i = j$$

$$\sum_{j=1}^{N}\left(K_{1ij} + \tfrac{1}{2}\delta_{ij}\right)\phi(Q_j) = \sum_{j=1}^{N} K_{2ij}\,\frac{\partial\phi(Q_j)}{\partial n}$$

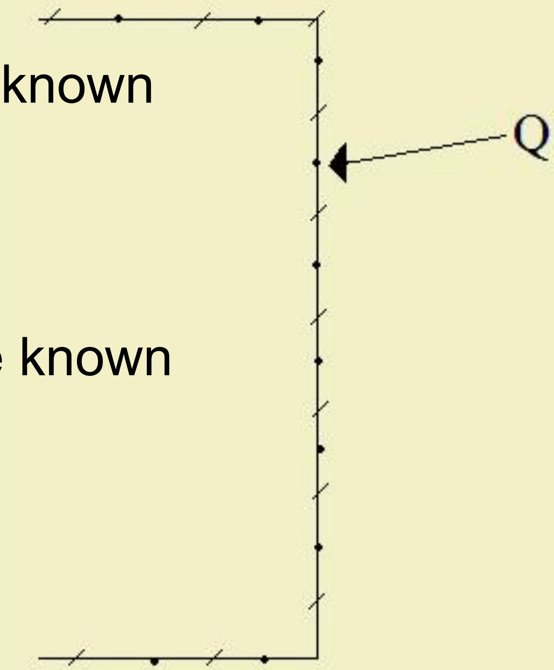
$$Ax = Bz$$

Numerical Implementation

Neumann Problem

$$Ax = c$$

Matrix A and vector C are known



Dirichlet Problem

$$c = Bz$$

Matrix B and vector C are known

Mixed case

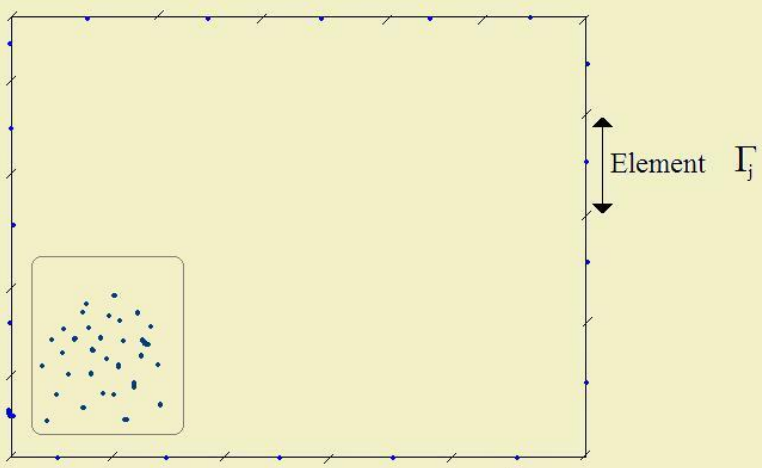
$$Ax = Bz$$

Unknowns and knowns can be separated in to same form as above

Numerical Implementation

As each point p in the domain is expressed in terms of the boundary values, once all boundary values are known ANY potential value within the domain can now be found.

$$C(P)\phi(P) = \int_\Gamma K_2(P,Q)\frac{\partial \phi}{\partial n}\, d\Gamma(Q) - \int_\Gamma K_1(P,Q)\phi(Q)\, d\Gamma(Q)$$



Boundary Element Method (BEM)

THE END

Book: The Boundary Element Method in Engineering A.A.BECKER