D Implementing parameter variability

The probability density function of random variable x that is Weibull distributed can be expressed as

$$\mathrm{We}(x) = \beta\alpha x^{\alpha-1}\exp(-\beta x^{\alpha}) \quad \text{for } x \geq 0 \,. \tag{18}$$

The expression used in the C++11 standard implementation is

$$\mathrm{We}(x) = \frac{a}{b}\left(\frac{x}{b}\right)^{a-1} \exp\left[-\left(\frac{x}{b}\right)^{a}\right] . \tag{19}$$

The relationship between these two expressions is

$$\alpha \equiv a \quad \text{and} \quad \beta \equiv \frac{1}{b^{a}} \implies b = \beta^{-1/\alpha} \,. \tag{20}$$

The shape parameter is $a = \alpha > 0$ and the scale parameter is $b > 0$. The shape parameter a is also called the **Weibull modulus** in the content of material strength distribution.

The mean of the distribution is

$$\mathbb{E}(x) = b\Gamma\left(1 + \frac{1}{a}\right) = \beta^{-1/\alpha}\Gamma\left(1 + \frac{1}{\alpha}\right) \tag{21}$$

where Γ is the gamma function. If we assume that the expected value is better represented by the median, we have

$$\mathbb{E}(x) = b\,[\ln(2)]^{1/a} \,. \tag{22}$$

The generate the Weibull distribution for a random variable, we typically use a transformation from a uniformly distributed random variable. To find the transformation between two probability distributions $f(y)$ and $g(x)$, we use the fundamental relation

$$f(y) = g(x)\left|\frac{dx}{dy}\right| \tag{23}$$

where the absolute value of the Jacobian of the transformation is used to make sure that probabilities sum to 1. For the special case where the distribution $g(x), x \in \mathcal{U} \sim [0,1]$ is uniform, we have

$$f(y) = \left|\frac{dx}{dy}\right| . \tag{24}$$

Therefore,

$$x = \int_0^y f(z)\,dz \,. \tag{25}$$

For the Weibull distribution, the right hand side is the cumulative distribution function,

$$x = \int_0^y \mathrm{We}(z)\,dz = \int_0^y \beta\alpha z^{\alpha-1}\exp(-\beta z^{\alpha})\,dz = 1 - \exp(-\beta y^{\alpha}) = 1 - \exp\left[-\left(\frac{y}{b}\right)^{a}\right] . \tag{26}$$

This relation can be inverted to give the transformed uniformly distributed random number between 0 and 1:

$$\boxed{y = \left[-\frac{1}{\beta}\ln(1-x)\right]^{1/\alpha} = b\,[-\ln(1-x)]^{1/a} \,.} \tag{27}$$

For a random variable that has the mean $\mathbb{E}(y) \approx \bar{y}$, from (21), the scale parameter is

$$b = \frac{\mathbb{E}(x)}{\Gamma\left(1 + \frac{1}{a}\right)} \approx \frac{\bar{y}}{\Gamma\left(1 + \frac{1}{a}\right)} \,. \tag{28}$$

Therefore, the Weibull-transformed uniformly distributed random variable can be written as

$$y = \frac{\bar{y}}{\Gamma\left(1+\frac{1}{a}\right)} \left[-\ln(1-x)\right]^{1/a} . \tag{29}$$

At this stage one typically invokes the fact that if x is uniformly distributed then so is $1-x$ and we can simplify the computation by using

$$\boxed{y = \frac{\bar{y}}{\Gamma\left(1+\frac{1}{a}\right)} \left[-\ln(x)\right]^{1/a} .} \tag{30}$$

Alternatively, we can assume that the sample median is a better approximation of the expected value and use equation (22) to compute the scale parameter:

$$b = \frac{\bar{y}}{\left[\ln(2)\right]^{1/a}} . \tag{31}$$

In that case we have

$$\boxed{y = \bar{y}\left[-\frac{\ln(x)}{\ln(2)}\right]^{1/a} = \bar{y}\left[\frac{\ln(x)}{\ln(1/2)}\right]^{1/a} .} \tag{32}$$

The existing implementation of the Weibull generator in Uintah uses the following approach. A uniformly distributed random number x is generated. This number is used to compute the quantity

$$F = \left[-\ln(x)\right]^{1/a} \tag{33}$$

where a is the Weibull modulus. Two other quantities are computed:

$$C = \left[\frac{v_{\text{expt}}}{v_{\text{elem}}}\right]^{1/m} \quad \text{and} \quad \eta = \frac{\bar{y}}{\Gamma\left(1+\frac{1}{a}\right)} \tag{34}$$

where v_{expt} is a reference volume, v_{elem} is the particle volume, m is an exponent, and $\bar{y}$ is the mean value of the parameter (y) that is Weibull distributed. The value of y is computed using the product of F, C, and η, giving

$$y = \left[\frac{v_{\text{expt}}}{v_{\text{elem}}}\right]^{1/m} \frac{\bar{y}}{\Gamma\left(1+\frac{1}{a}\right)} \left[-\ln(x)\right]^{1/a} \tag{35}$$

The code typically uses $m = a$ to get

$$\boxed{y = \frac{\bar{y}}{\Gamma\left(1+\frac{1}{a}\right)} \left[-\frac{v_{\text{expt}}}{v_{\text{elem}}} \ln(x)\right]^{1/a} .} \tag{36}$$

This expression is identical to equation (30) except for a size-effect factor. Note that (32) is the form used in Scott Swan's thesis (previously implemented in Uintah):

$$y = \left[\frac{v_{\text{expt}}}{v_{\text{elem}}}\right]^{1/a} \bar{y}\left[\frac{\ln x}{\ln(1/2)}\right]^{1/a} . \tag{37}$$

For our purposes, if we use the C++11 Weibull distribution generator, we can incorporate the volume scaling by just multiplying the scaling factor to the number generated, i.e.,

$$y = \left[\frac{v_{\text{expt}}}{v_{\text{elem}}} \right]^{1/m} \text{We}(\bar{y}, a, b, R) \tag{38}$$

where R is the uniformly distributed pseudorandom number in $[0, 1]$ generated by the Mersenne twister algorithm.