Mathematics of Imaging *Xiang Hao*

Assignment01

1) Consider the rocket example 3 page 125. What is the solution to the optimization problem if L^2 norm of u(t) is minimized instead.

Solution 1:

$F(u) = \int_0^T |u(t)|^2 dt + \lambda(\int_0^T (T-t)u(t)dt - (1+\frac{T^2}{2}))$

$F(u+\epsilon h) = \int_0^T |u(t)+\epsilon h(t)|^2 dt + \lambda(\int_0^T (T-t)(u(t)-\epsilon h(t))dt - (1+\frac{T^2}{2}))$

$\frac{dF(u+\epsilon h)}{d\epsilon} = \int_0^T 2(u(t)+\epsilon h(t))h(t)dt + \lambda(\int_0^T (T-t)h(t)dt)$

$\frac{dF(u+\epsilon h)}{d\epsilon}|_{\epsilon=0} = \int_0^T 2u(t)h(t)dt + \lambda(\int_0^T (T-t)h(t)dt) = \int_0^T (2u(t)+\lambda(T-t))h(t)dt$

So, $2u(t)+\lambda(T-t) = 0$, and $u(t) = -\frac{\lambda}{2}(T-t)$.

Since, $\int_0^T (T-t)u(t)dt = 1+\frac{T^2}{2}$, we get

$\int_0^T (T-t)u(t)dt = -\int_0^T (T-t)\frac{\lambda}{2}(T-t)dt = -\frac{\lambda}{2}\int_0^T T^2 - 2Tt + t^2 dt = -\frac{\lambda}{2}(T^2t - Tt^2 + \frac{1}{3}t^3)|_0^T = -\frac{\lambda}{2}(T^3 - T^3 + \frac{1}{3}T^3) = -\frac{\lambda}{6}T^3 = 1+\frac{T^2}{2}$

So, $\lambda = -\frac{6+3T^2}{T^3}$

So, $u(t) = \frac{6+3T^2}{2T^3}(T-t)$

Solution 2:

The L^2 norm of u(t) is $(\int_0^T |u(t)|^2 dt)^{\frac{1}{2}}$. We want to minimize it with constraint $\int_0^T (T-t)u(t)dt = 1+\frac{T^2}{2}$.

If u is any function satisfying the constraint, $\min_{<T-t,u>=1+\frac{T^2}{2}} ||u||_2 = \min_{m^* \in M^\perp} ||u-m^*||_2$, where M denotes the space generated by T-t, a subspace of $L_2[0,T]$.

From theorem 2 of the book, on page 121,

$\min_{m^* \in M^\perp} ||u-m^*||_2 = \sup_{m\in M, ||x||_2\le 1} < m, u >= \max_{||(T-t)a||_2\le 1}(1+\frac{T^2}{2})a$

$||(T-t)a||_2 = (\int_0^T ((T-t)a)^2 dt)^{\frac{1}{2}} = (a^2(T^2 - \frac{1}{2}T^2))^{\frac{1}{2}} = \frac{\sqrt{2}}{2}T|a|$

Since $||(T-t)a||_2 \le 1$, we get $|a| \le \frac{\sqrt{2}}{T}$

So, $\min_{<T-t,u>=1+\frac{T^2}{2}} ||u||_2 = \max_{||(T-t)a||_2\le 1}(1+\frac{T^2}{2})a = \max_{|a|\le\frac{\sqrt{2}}{T}}(1+\frac{T^2}{2})a = (1+\frac{T^2}{2})\frac{\sqrt{2}}{T} = \frac{\sqrt{2}}{T} + \frac{\sqrt{2}T}{2}$

$(\frac{\sqrt{2}}{T} + \frac{\sqrt{2}T}{2})_T = \sqrt{2}(-1)\frac{1}{T^2} + \frac{\sqrt{2}}{2} = 0$, we get $T = \sqrt{2}$

So, $\min_{<T-t,u>=1+\frac{T^2}{2}} ||u||_2 = 2$.

The optimal u must be aligned with $(T-t)\frac{\sqrt{2}}{T}$.

2) Each student is asked to pick a problem from Chapters 4 or 5 that interests them most and solve it. Please post your solutions to the wiki so that your colleagues can comment on them.

Question 12 on Page 139

$L_2(x) = \sum_{k=0}^2 a_{2k}x(\frac{k}{2}) = a_{20}x(0) + a_{21}x(0.5) + a_{22}x(1)$

Let $p = a + bt + ct^2$

$L(p) = \int_0^1 P(t)dt = \int_0^1 a + bt + ct^2 dt = (\frac{1}{3}ct^3 + \frac{1}{2}bt^2 + at)|_0^1 = \frac{1}{3}c + \frac{1}{2}b + a.$

$L_2(p) = a_{20}a + a_{21}(a + \frac{1}{2}b + \frac{1}{4}c) + a_{22}(a + b + c) = (a_{20} + a_{21} + a_{22})a + (\frac{a_{21}}{2} + a_{22})b + (\frac{1}{4}(a_{21} + a_{22})c$

Since $L(p) = L_2(p)$, we get $a_{20} + a_{21} + a_{22} = 1$, $\frac{a_{21}}{2} + a_{22} = \frac{1}{2}$, $\frac{a_{21}}{4} + a_{22} = \frac{1}{3}$

So, $a_{20} = \frac{1}{6}, a_{21} = \frac{2}{3}, a_{22} = \frac{1}{6}$.

So, $L_2(x) = \frac{1}{6}x(0) + \frac{2}{3}x(0.5) + \frac{1}{6}x(1)$

3) Question 21 on page 76

Q can be decomposed in to $L^T L$ since Q is a SPD matrix, then $x^T Q x = x^T L^T L x = (Lx)^T Lx$. Let y=Lx, then $x = L^{-1}y$. Let $B = AL^{-1}$, then the problem can be restated as:

minimize the norm of y subject to $By = b$.

Suppose y_0 has the minumum norm, $y_0 = (\beta B)^T$, β is a 1xm matrix

Based on the projection theorem,

$B(y - y_0) = 0$
$By_0 = By = b$
$B(\beta B)^T = b$
$BB^T \beta^T = b$
$\beta^T = (BB^T)^{-1}b = (AL^{-1}(AL^{-1})^T)^{-1}b$