CS 6220 Spring 2011
Homework Set Four
Interpolation and Integration Methods

Runge's function is $f(x) = 1/(1 + 25x^2)$, $x \in [-1, 1]$.

The integral $I = \int_{-1}^{1} f(x)dx$ takes on the value $2\tan^{-1}(5) \approx 2.746801533890031788$.

I. Interpolation

1. Compute the Newton interpolating polynomial p_n for $f(x)$ for $n = 5, 10, 15$ using the divided difference algorithm with:

(a) equally spaced nodes;

(b) Chebyshev nodes.

2. Create a table that shows all three values of $max_{0 \le k \le 30} |f(x_k) - p_n(x_k)|$,
$n = 5, 10, 15$, with equally spaced points $x_k = -1 + k/15$.

3. Create a graph that displays f, p_5, p_{10} and p_{15}.

4. Briefly discuss your results.

Here is the divided difference algorithm.

```
for i=0 to n do
    d_i=f(x_i)
end do
for j=1 to n do
      for i=n to j step -1 do
           d_i=(d_i-d_{i-1})/(x_i-x_{i-j})
      end do
end do
```

II. Cubic Splines

1. Determine the natural cubic spline $S(x)$ for $f(x)$ and equally spaced points $x_0 = -1, x_1, , \dots, x_n = 1$ on $[-1, 1]$ for $n = 5, 10, 15$.

On each subinterval $[x_i, x_{i+1}]$ function S equals function S_i where

$$S_i(x) = \frac{z_i}{6h_i}(x_{i+1} - x)^3 + \frac{z_{i+1}}{6h_i}(x - x_i)^3 + \left(\frac{f(x_{i+1})}{h_i} - \frac{z_{i+1}h_i}{6}\right)(x - x_i) + \left(\frac{f(x_i)}{h_i} - \frac{z_i h_i}{6}\right)(x_{i+1} - x)$$

with the z_i solution to

$$\begin{bmatrix} u_1 & h_1 & & & & \\ h_1 & u_2 & h_2 & & & \\ & h_2 & u_3 & h_3 & & \\ & & \ddots & & \ddots & \\ & & & h_{n-3} & u_{n-2} & h_{n-2} \\ & & & & h_{n-2} & u_{n-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-2} \\ z_{n-1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-2} \\ v_{n-1} \end{bmatrix}$$

and where $h_i = x_{i+1} - x_i$, $u_i = 2(h_i + h_{i-1})$, $b_i = (f(x_{i+1}) - f(x_i))/h_i$, $v_i = 6(b_i - b_{i-1})$.

Here is pseudocode you can use:

```
input n, {x_i}, {f(x_i)}
for i=0 to n-1 do
     h_i=x_{i+1}-x_i
     b_i=6(f(x_{i+1})-f(x_i))/h_i
end do
u_1=2(h_0+h_1)
v_1=b_1-b_0
for i=2 to n-1 do
     u_i=2(h_i+h_{i-1})-h_{i-1}^2/u_{i-1}
     v_i=b_i-b_{i-1}-h_{i-1}v_{i-1}/u_{i-1}
end do
z_n=0
for i=n-1 to 1 step -1 do
     z_i=(v_i-h_iz_{i+1})/u_i
end do
z_0=0
output {z_i}
```

2. Create a table that shows all three values of $max_{0\leq k\leq 30}|f(x_k) - S(x_k)|$, $n = 5, 10, 15$, with equally spaced points $x_k = -1 + k/15$.

3. Create a graph that displays f and the three spline functions found in 1.

4. Briefly discuss your results.

III. Interpolatory Integration

1. By "inverting" the 4x4 Vandermonde matrix determine $V(x_0, x_1, x_2, x_3)$ determine the coefficients w_0, w_1, w_2, w_3 such that

$$\sum_{i=0}^{3} w_i p\left(x_i\right) = \int_{-1}^{1} p(x)dx$$

holds for $p(x) = 1, x, x^2, x^3$ when

(a) $x_0 = -1, x_1, x_2, x_3 = 1$ are equally spaced points on $[-1, 1]$.

(b) x_0, x_1, x_2, x_3 are the roots of $T_4(x)$, the 4^{th} Chebyshev polynomial.

2. Using your results from 1., compute values for the integral I, and then compare with $2\tan^{-1}(5)$.

IV. Newton-Cotes Integration

If we write the Newton-Cotes formula for I as

$$I \approx \frac{2}{n}\sum_{k=0}^{n} w_{n,k} f(-1 + 2k/n)$$

we know from lecture that

$$w_{n,k} = w_{n,n-k} = \frac{(-1)^{n-k}}{k!(n-k)!}\int_{0}^{n} t(t-1)\cdots(t-n)/(t-k)\,dt.$$

With the help of either Maple or Matlab, compute enough of the above $w_{n,k}$ to come up with Newton-Cotes approximations for I in the cases $n = 5, 10, 15$. Now compare with $2\tan^{-1}(5)$.

V. Gaussian Quadrature

The Gaussian Quadrature formula

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

allows us to choose the $n+1$ nodes and $n+1$ coefficients A_i at will with no *a priori* restrictions on them. By choosing them judiciously the above approximation becomes equality when Runge's function f is replaced by any polynomial of degree $2n+1$ or less.

Here are a few of the values:

$$n = 1 : A_0 = A_1 = 1\,, -x_0 = x_1 = 1/\sqrt{3} \approx .57735026918962576$$

$$n = 2 : A_0 = A_2 = 5/9\,, A_1 = 8/9\,, x_1 = 0\,, -x_0 = x_2 = \sqrt{.6} \approx .7745966692414834$$

$$n = 4 : A_0 = A_4 = .3(.7 + 5\sqrt{.7})/(2 + 5\sqrt{.7}) \approx .236926885056189$$

$$A_1 = A_3 = .3(-.7 + 5\sqrt{.7})/(-2 + 5\sqrt{.7}) \approx .478628670499366$$

$$A_2 = 128/225 \approx .568888888888889$$

$$-x_0 = x_4 = \frac{1}{3}\sqrt{5 + 2\sqrt{10/7}} \approx .906179845938664$$

$$-x_1 = x_3 = \frac{1}{3}\sqrt{5 - 2\sqrt{10/7}} \approx .538469310105683$$

$$x_2 = 0$$

Using these values compute three Gaussian Quadrature approximations to I, and then compare with $2\tan^{-1}(5)$.

VI. Compare and discuss all the above results as you see fit. (Strive for conciseness and insight as much as possible.)