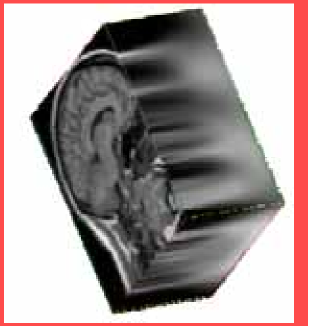

Lecture 2: FEV 3,4: Gaussian Kernel, Gaussian Derivatives

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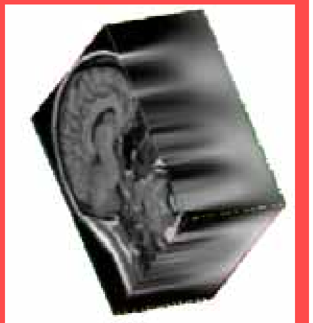
CS 7960, Spring 2010



Aperture function: Operator

- Unconstrained front-end: Unique solution to aperture function is Gaussian kernel
- Many derivations, all leading to Gaussian kernel

$\sigma = 1; \text{Plot}\left[\frac{1}{\sqrt{2\,\pi\,\sigma^2}}\,E^{-\frac{x^2}{2\,\sigma^2}},\ \{x, -4, 4\},\ \text{ImageSize} \text{ ->}\right.$

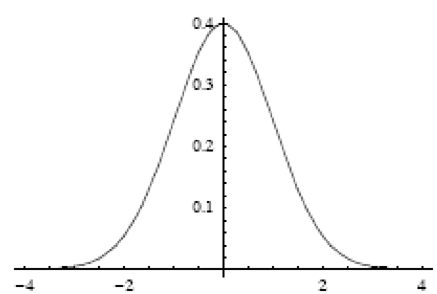
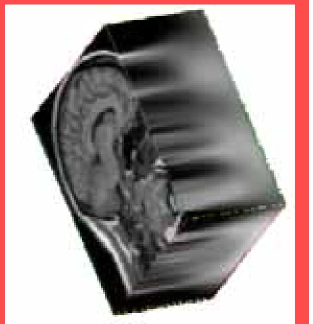


Figure 2.1 The Gaussian kernel with unit standard deviation in 1D.



Gaussian Kernel

$$G_{1D}(x;\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{x^2}{2\sigma^2}},\; G_{2D}(x,y;\sigma) = \frac{1}{2\pi\sigma^2}\, e^{-\frac{x^2+y^2}{2\sigma^2}},\; G_{ND}(\vec{x};\sigma) = \frac{1}{\left(\sqrt{2\pi}\,\sigma\right)^N}\, e^{-\frac{|\vec{x}|^2}{2\sigma^2}}$$

Normalization to 1.0

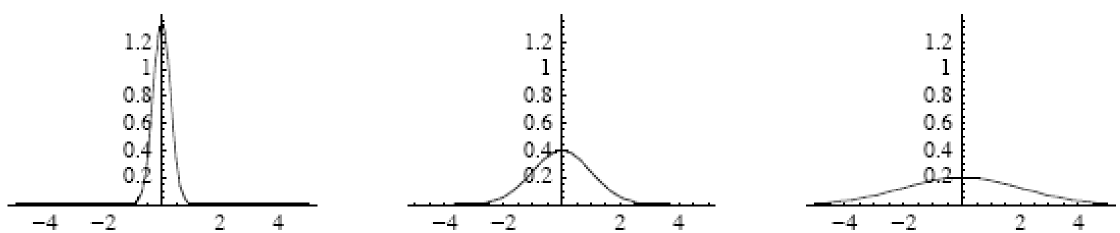
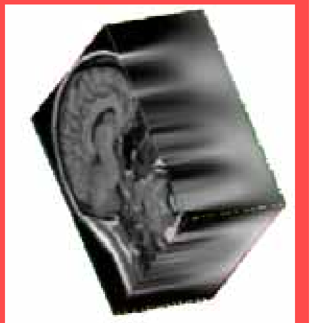


Figure 3.2 The Gaussian function at scales $\sigma = .3$, $\sigma = 1$ and $\sigma = 2$. The kernel is normalized, so the total area under the curve is always unity.



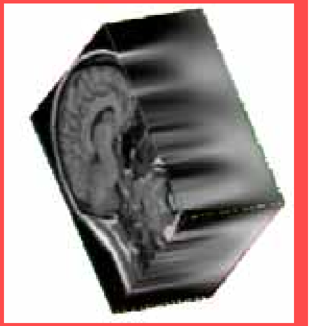
Cascade Property, Self Similarity

- Convolution of 2 Gaussian kernel produces another Gaussian kernel
- Squares of σ's add up

$$g_{\text{new}}(\vec{x};\sigma_1^2+\sigma_2^2) = g_1(\vec{x};\sigma_1^2) \otimes g_2(\vec{x};\sigma_2^2)$$

$$\sigma =.;\ \text{Simplify}\left[\int_{-\infty}^{\infty} \text{gauss}[\alpha, \sigma_1]\, \text{gauss}[\alpha - x, \sigma_2]\, d\alpha,\ \{\sigma_1 > 0, \sigma_2 > 0\}\right]$$

$$\frac{e^{-\frac{x^2}{2(\sigma_1^2+\sigma_2^2)}}}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}}$$



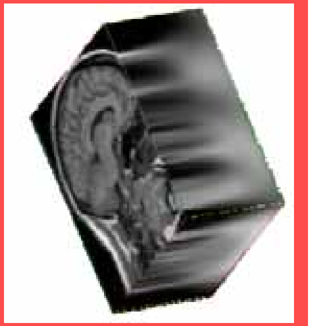
Scale Property

- Gaussian kernel is Green's function of linear, isotropic diffusion equation

$$g := \frac{1}{2\pi\sigma^2}\,\mathrm{Exp}\left[-\frac{x^2+y^2}{2\sigma^2}\right];$$

$$\frac{\partial^2 L}{\partial x^2} + \frac{\partial^2 L}{\partial y^2} = L_{\mathrm{xx}} + L_{\mathrm{yy}} = \frac{\partial L}{\partial t}$$

$$t = 2\,\sigma^2$$



Cumulative Gaussian Fct.

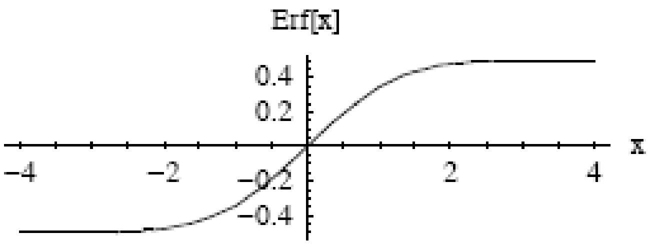
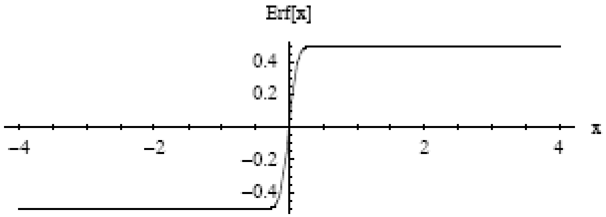


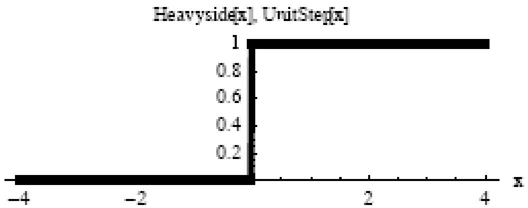
Figure 3.4 The error function **Erf[x]** is the cumulative Gaussian function.

Gaussian vs. Dirac

- Gaussian function is blurred version of the Delta Dirac function.
- Cumulative Gaussian function (Erf) is blurred version of the Heavyside step edge.

$$\lim_{\sigma \downarrow 0} \left(\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} \right) = \delta(x)$$

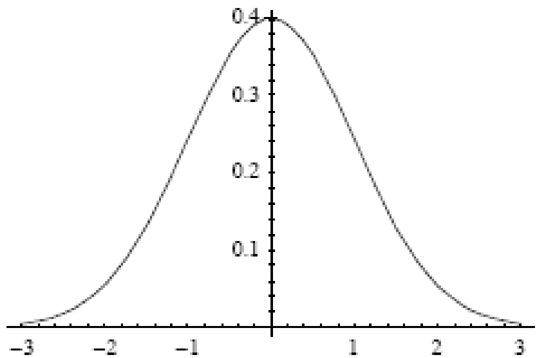




Separability

$$g_{2D}(x, y; \sigma_1^2 + \sigma_2^2) = g_{1D}(x; \sigma_1^2) \otimes g_{1D}(y; \sigma_2^2)$$

```
DisplayTogetherArray[{Plot[gauss[x, σ = 1], {x, -3, 3}],
   Plot3D[gauss[x, σ = 1] gauss[y, σ = 1], {x, -3, 3}, {y, -3, 3}]},
  ImageSize -> 440];
```



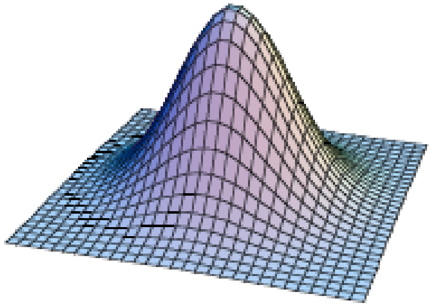
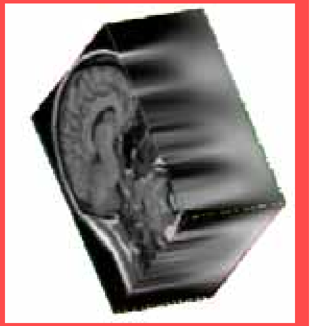


Figure 3.7 A product of Gaussian functions gives a higher dimensional Gaussian function. This is a consequence of the separability.



Relation to binomial coefficients

```
Expand[(x + y)^30]
```

```
ListPlot[Table[Binomial[30, n], {n, 1, 30}],
  PlotStyle -> {PointSize[.015]}, AspectRatio -> .3];
```

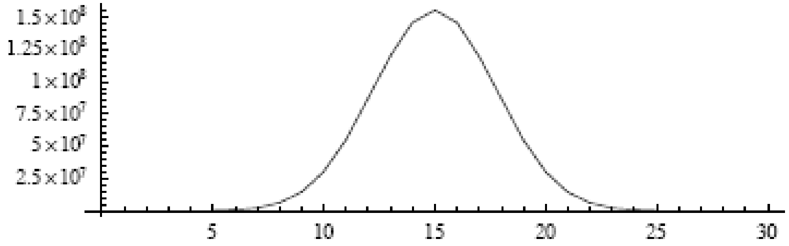
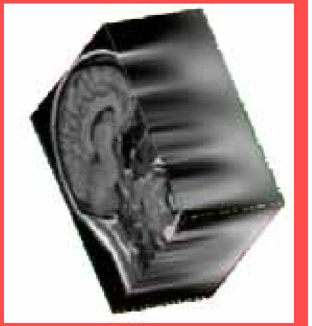


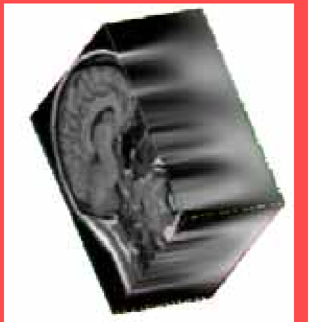
Figure 3.8 Binomial coefficients approximate a Gaussian distribution for increasing order.



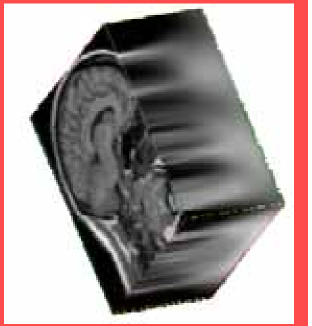
Diffusion Equation

- Gaussian is the solution of the linear diffusion equation (with: $t = 2\sigma^2$)
- Intensity is diffused over time (scale) in all directions isotropically

$$\frac{\partial L}{\partial t} = \frac{\partial^2 L}{\partial x^2} + \frac{\partial^2 L}{\partial y^2} = \Delta L$$



FEV 4: Gaussian Derivatives



Derivatives

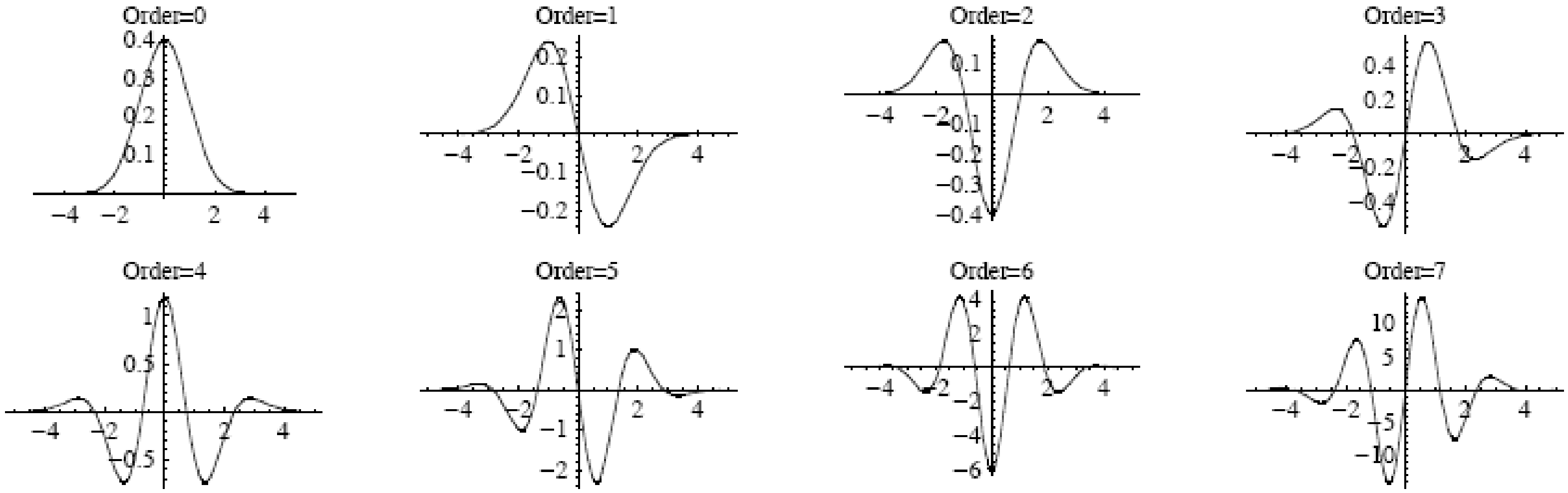
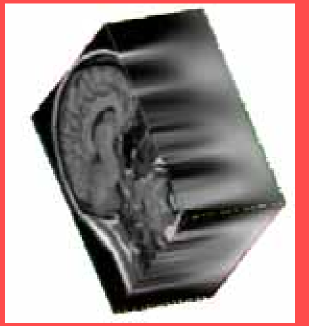
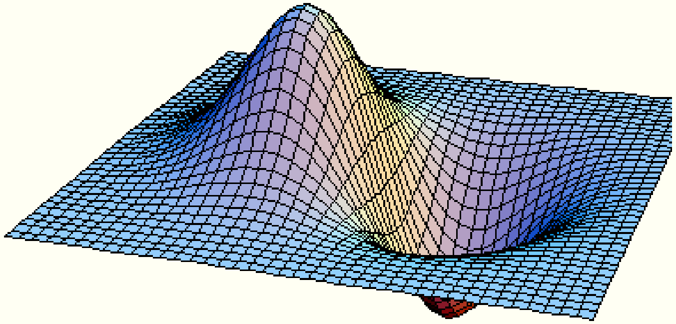


Figure 4.1 Plots of the 1D Gaussian derivative function for order 0 to 7.

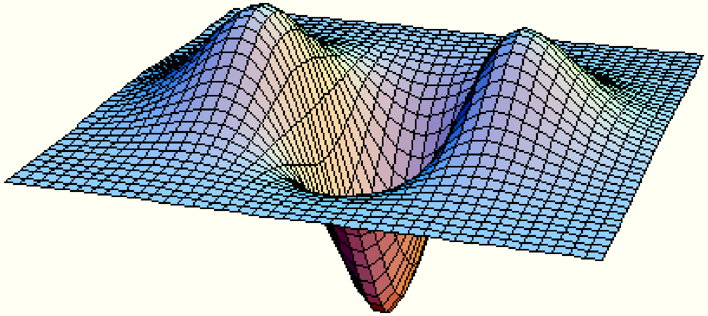


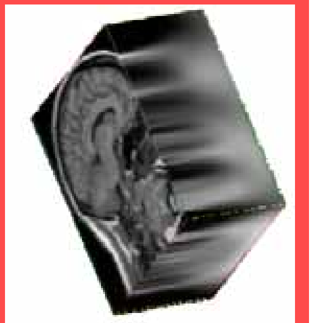
Higher Dimensions

x: 1st y: 0



x: 2nd y: 0





Other Families: Gabor Filters

$$\text{gabor}[x\_,\ \sigma\_] := \text{Sin}[x]\ \frac{1}{\sqrt{2\,\pi\,\sigma^2}}\ \text{Exp}\left[-\frac{x^2}{2\,\sigma^2}\right];$$