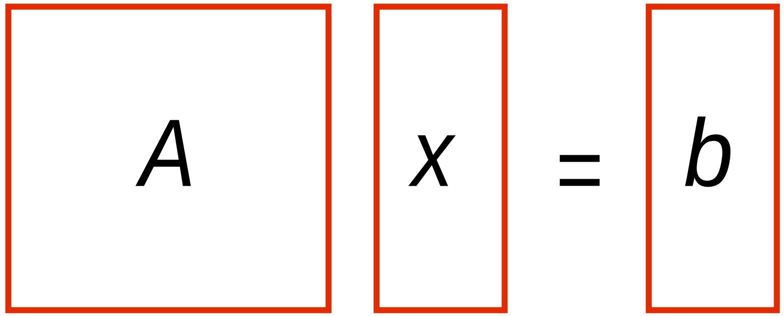
Solution of Linear Systems

Application: To be used for calculation of linear transformation based on sets of landmarks, e.g.

Materials and Matlab

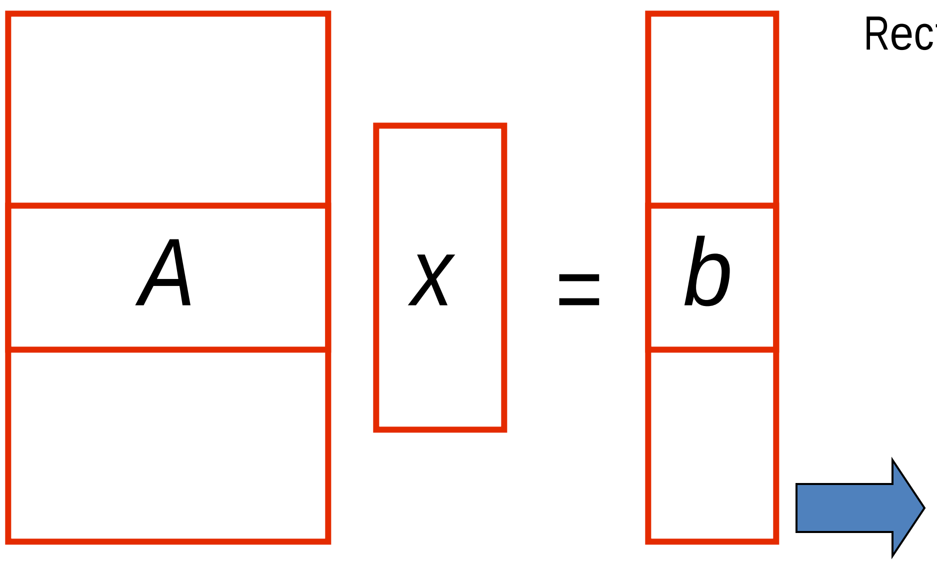
- http://audition.ens.fr/brette/calculscientifique/lecture6.pdf
- http://en.wikipedia.org/wiki/Overdetermined_system
- http://www.mathworks.com/help/toolbox/optim/ug/brhkghv-18.html
- http://www.mathworks.com/help/techdoc/math/f4-2224.html#f4-2282

Linear Systems



Square system:

- unique solution
- Gaussian elimination



Rectangular system ??

- underconstrained:
  infinity of solutions
- overconstrained:
  no solution

Minimize $|Ax-b|^2$

How do you solve overconstrained linear equations ??

- Define $E = |\boldsymbol{e}|^2 = \boldsymbol{e} \cdot \boldsymbol{e}$ with

$$\boldsymbol{e} = A\boldsymbol{x} - \boldsymbol{b} = \left[\begin{array}{c|c|c|c} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \ldots & \boldsymbol{c}_n \end{array} \right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \boldsymbol{b}$$

$$= x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots x_n\boldsymbol{c}_n - \boldsymbol{b}$$

- At a minimum,

$$\frac{\partial E}{\partial x_i} = \frac{\partial \boldsymbol{e}}{\partial x_i} \cdot \boldsymbol{e} + \boldsymbol{e} \cdot \frac{\partial \boldsymbol{e}}{\partial x_i} = 2\frac{\partial \boldsymbol{e}}{\partial x_i} \cdot \boldsymbol{e}$$

$$= 2\frac{\partial}{\partial x_i}(x_1\boldsymbol{c}_1 + \cdots + x_n\boldsymbol{c}_n - \boldsymbol{b}) \cdot \boldsymbol{e} = 2\boldsymbol{c}_i \cdot \boldsymbol{e}$$

$$= 2\boldsymbol{c}_i^T(A\boldsymbol{x} - \boldsymbol{b}) = 0$$

- or

$$0 = \begin{bmatrix} \boldsymbol{c}_i^T \\ \vdots \\ \boldsymbol{c}_n^T \end{bmatrix} (A\boldsymbol{x} - \boldsymbol{b}) = A^T(A\boldsymbol{x} - \boldsymbol{b}) \Rightarrow A^TA\boldsymbol{x} = A^T\boldsymbol{b},$$

where $\boxed{\boldsymbol{x} = A^\dagger\boldsymbol{b} \text{ and } A^\dagger = (A^TA)^{-1}A^T \text{ is the } \textit{pseudoinverse} \text{ of } A\text{ !}}$

Overconstrained Problems in Matlab

Problem: solve A for X*A=Y:

- if full rank: unique solution
- if overconstrained: This means we want to find the solution that minimizes \sum_{(x,y) pairs} (y-xA)^2

- **Solution 1**: Left Matrix Divide: **A=X\Y** is the matrix division of X into Y, which is roughly the same as INV(X)*Y , except it is computed in a different way.
- **Solution 2**: Use pseudoinverse: **X=A⁺*Y**. The pseudoinverse is calculated as: A = Y*pinv(X). However, it may be easier to write out the system as X*A=Y and then do A = X \ Y (solution 1) which is pretty standard.