Fourier Transform in Image Processing

CS/BIOEN 6640 U of Utah

Guido Gerig

(slides modified from

Marcel Prastawa 2012)

Basis Decomposition

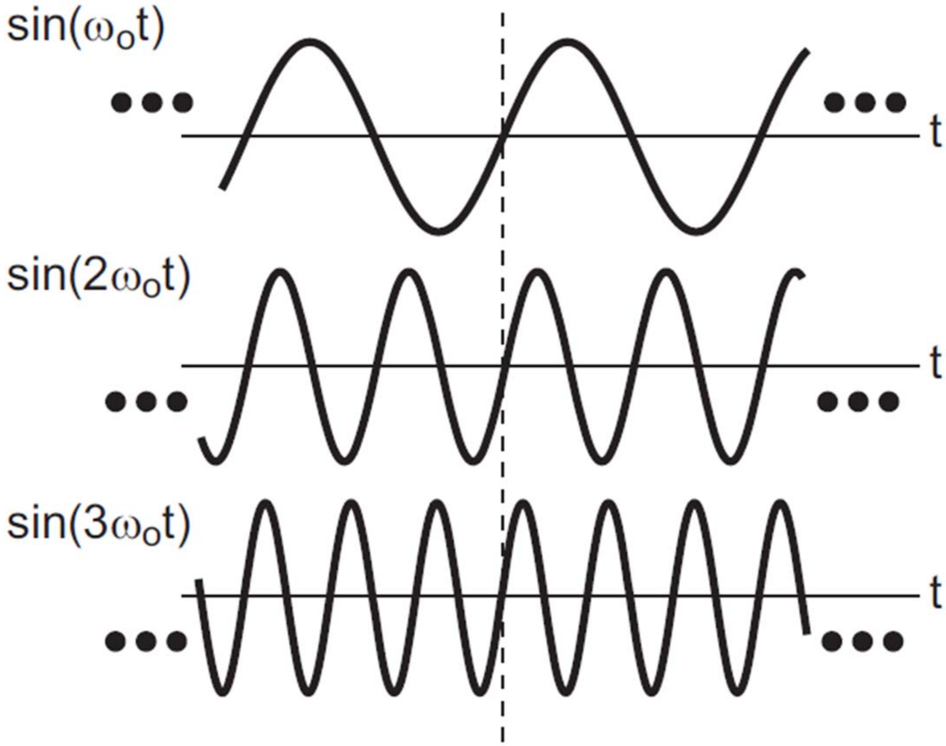
- Write a function as a weighted sum of basis functions

$$f(x) = \Sigma w_i B_i(x)$$

- What is a good set of basis functions?
- How do you determine the weights?

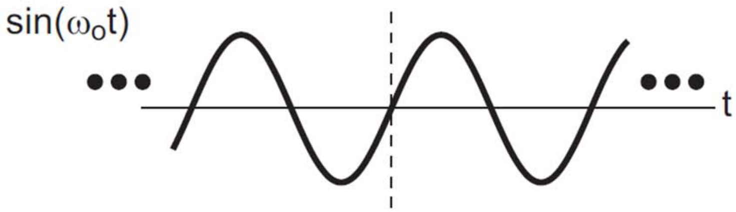
Sine Waves

- Use sine waves of different frequencies as basis functions?



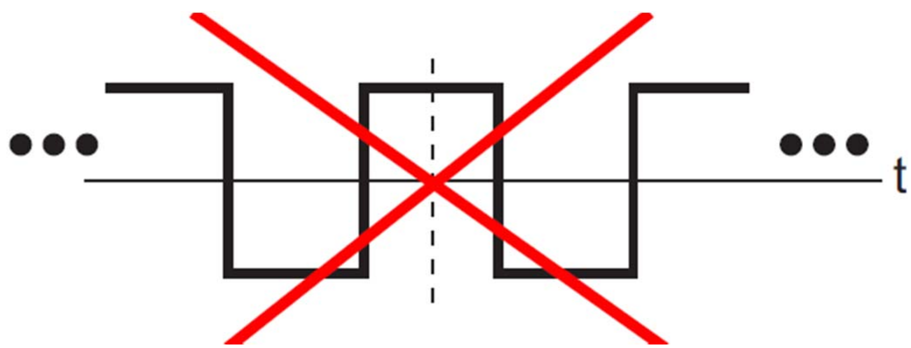
Limitation of Sines

- Sines are odd / anti-symmetric:



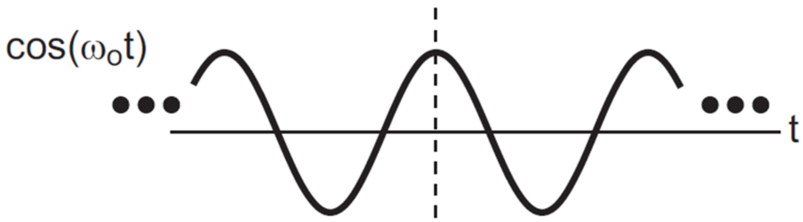
$$f(t) = -f(-t)$$

- Sine basis cannot create even functions:



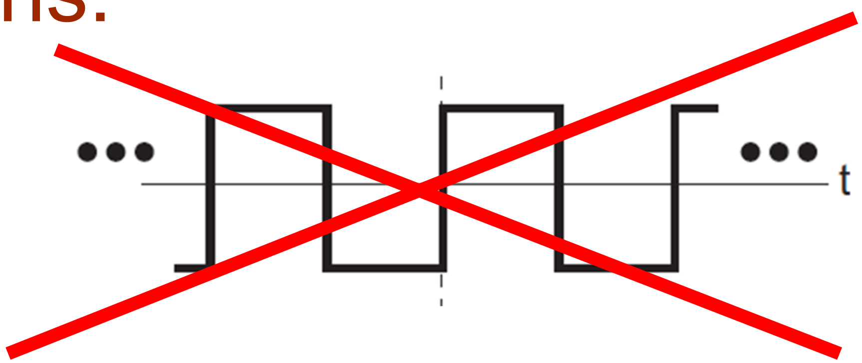
Limitation of Cosines

- Cosines are even / symmetric functions:



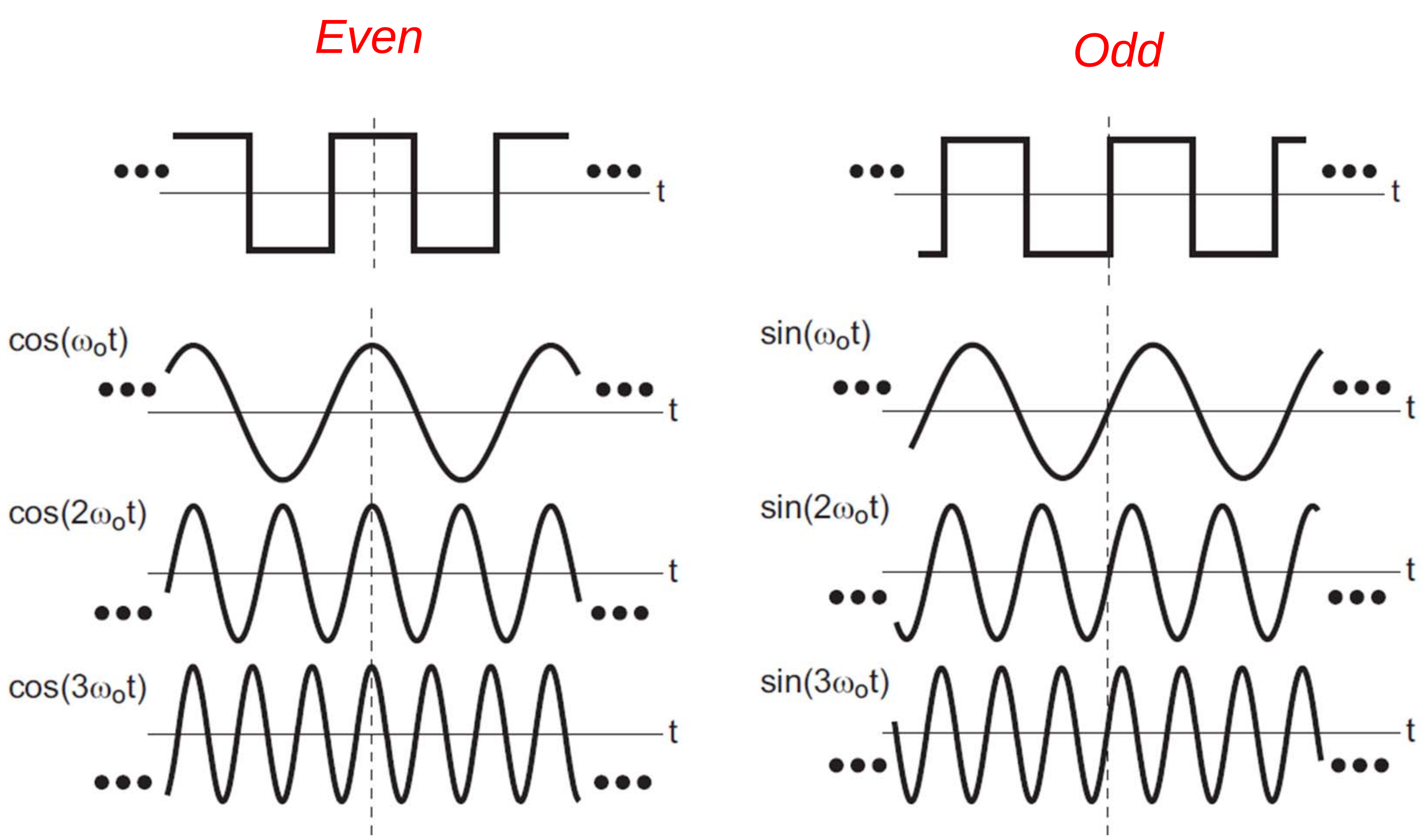
$$f(t) = f(-t)$$

- Cosine basis cannot create odd functions:



Combine Cosines and Sines

- Allow creation of both even and odd functions with different combinations:



Why Sines and Cosines?

- Represent functions as a combination of basis with different frequencies

Why Sines and Cosines?

- Represent functions as a combination of basis with different frequencies
- Intuitive description of signals / images:
  - how much high frequency content?
  - what do the low freq. content look like?

Why Sines and Cosines?

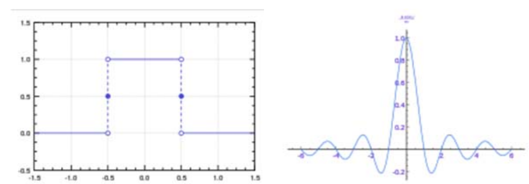
- Represent functions as a combination of basis with different frequencies
- Intuitive description of signals / images:
  - how much high frequency content?
  - what do the low freq. content look like?
- Image processing "language":
  - remove noise by reducing high freq content
  - explains sampling / perception phenomena

Jean Baptiste Joseph Fourier

Basic contributions 1807:

- **Fourier Series**: Represent any periodic function as a weighted combination of sine and cosines of different frequencies.

- **Fourier Transform**: Even non-periodic functions with finite area: Integral of weighted sine and cosine functions.



- Functions (signals) can be completely reconstructed from the Fourier domain without loosing any information.

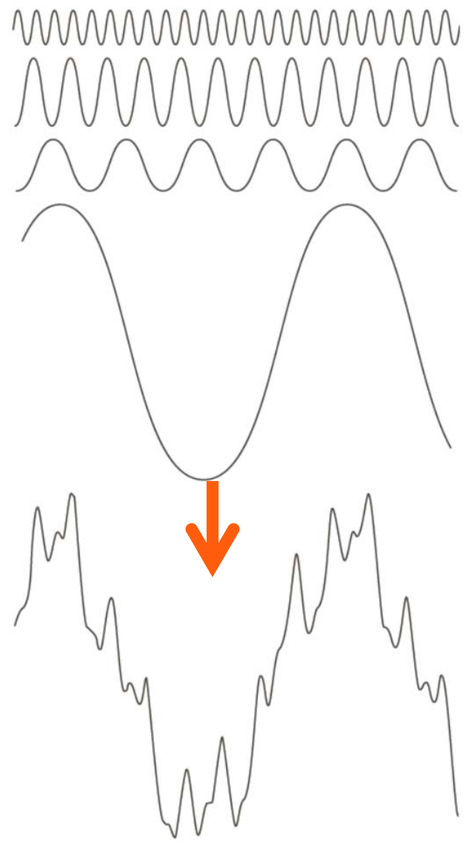


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

The Fourier Transform

Reminder: Euler's Identity

- From calculus

$$e^{jx} = \cos x + j\sin x$$

- j is the imaginary part of a complex number

Fourier Transform

- Forward, mapping to frequency domain:

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt$$

- Backward, inverse mapping to time domain:

$$f(t) = \int_{-\infty}^{\infty} F(s) e^{+j2\pi st} ds$$

Space and Frequency

Fourier Transform

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

Fourier Synthesis

Fourier Analysis

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

Inverse Fourier Transform

Fourier Series

- Projection or change of basis
- Coordinates (coeffs) in Fourier basis:

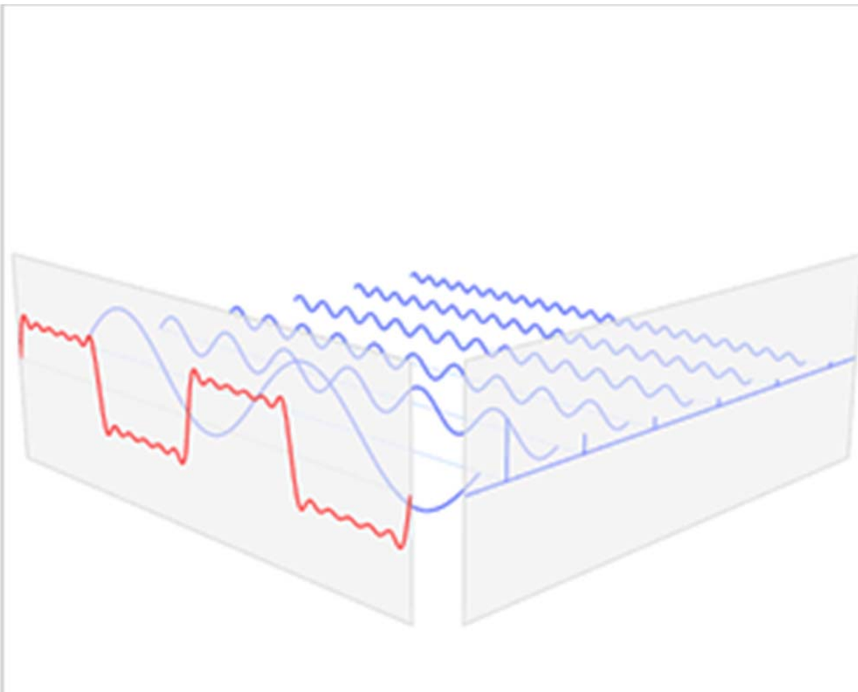
$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-j\frac{2\pi n}{T}t} dt$$

- Rewrite f as:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \sin\left(j2\pi\frac{n}{T}t\right) + \sum_{n=1}^{\infty} b_n \cos\left(j2\pi\frac{n}{T}t\right)$$

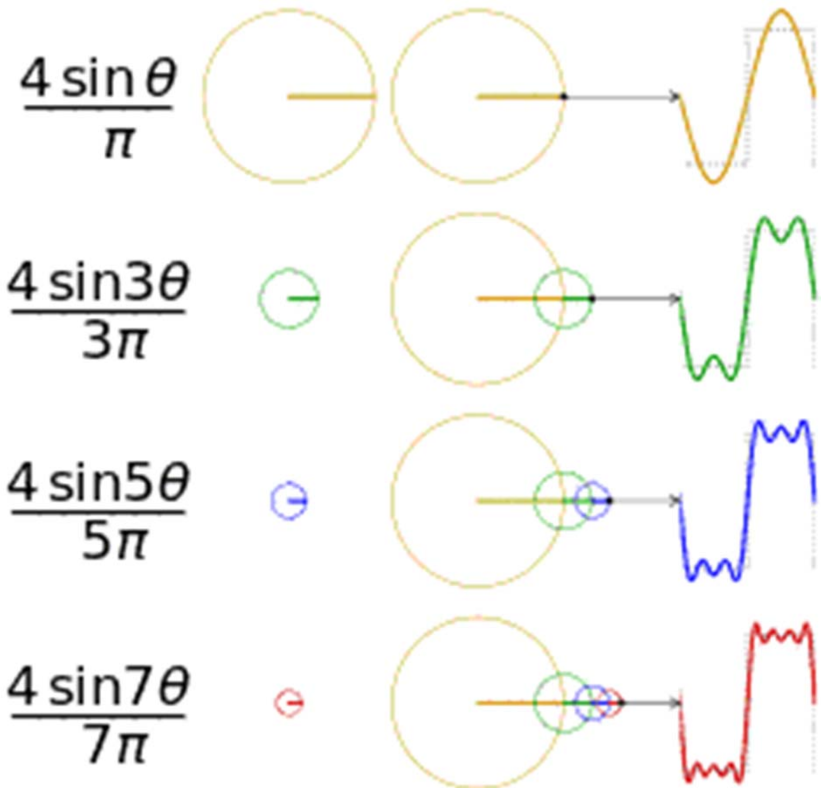
Fourier Series: f(t) →F(s)



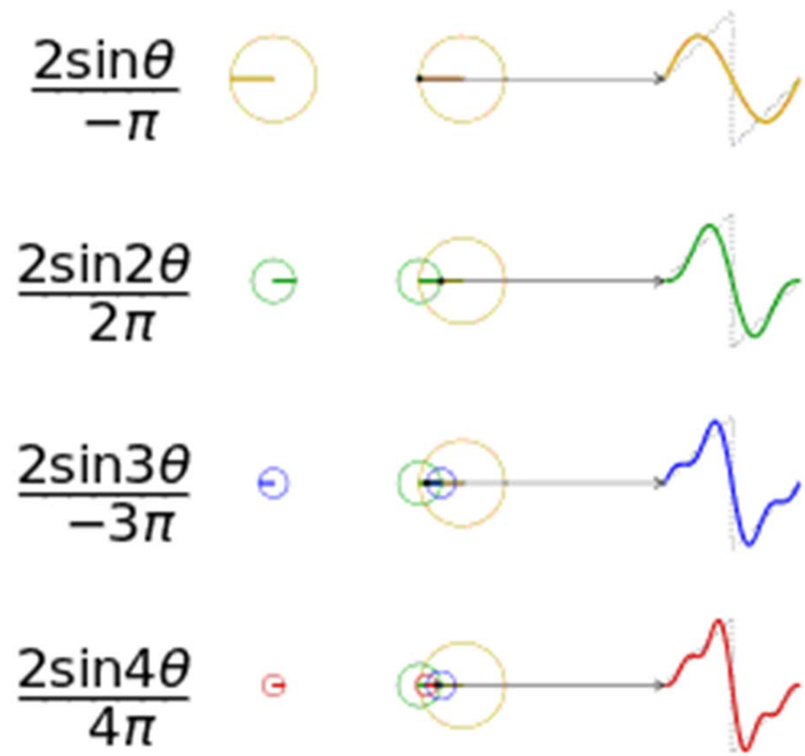
Function *s*(*x*) (in red) is a sum of six sine functions of different amplitudes and harmonically related frequencies. Their summation is called a Fourier series. The Fourier transform, *S*(*f*) (in blue), which depicts amplitude vs frequency, reveals the 6 frequencies and their amplitudes.

source: http://en.wikipedia.org/wiki/Fourier_series

Fourier Series: Approximation



Visualisation of an approximation of a square wave by taking the first 1, 2, 3 and 4 terms of its Fourier series



Visualisation of an approximation of a sawtooth wave of the same amplitude and frequency for comparison

Source: http://en.wikipedia.org/wiki/Fourier_series

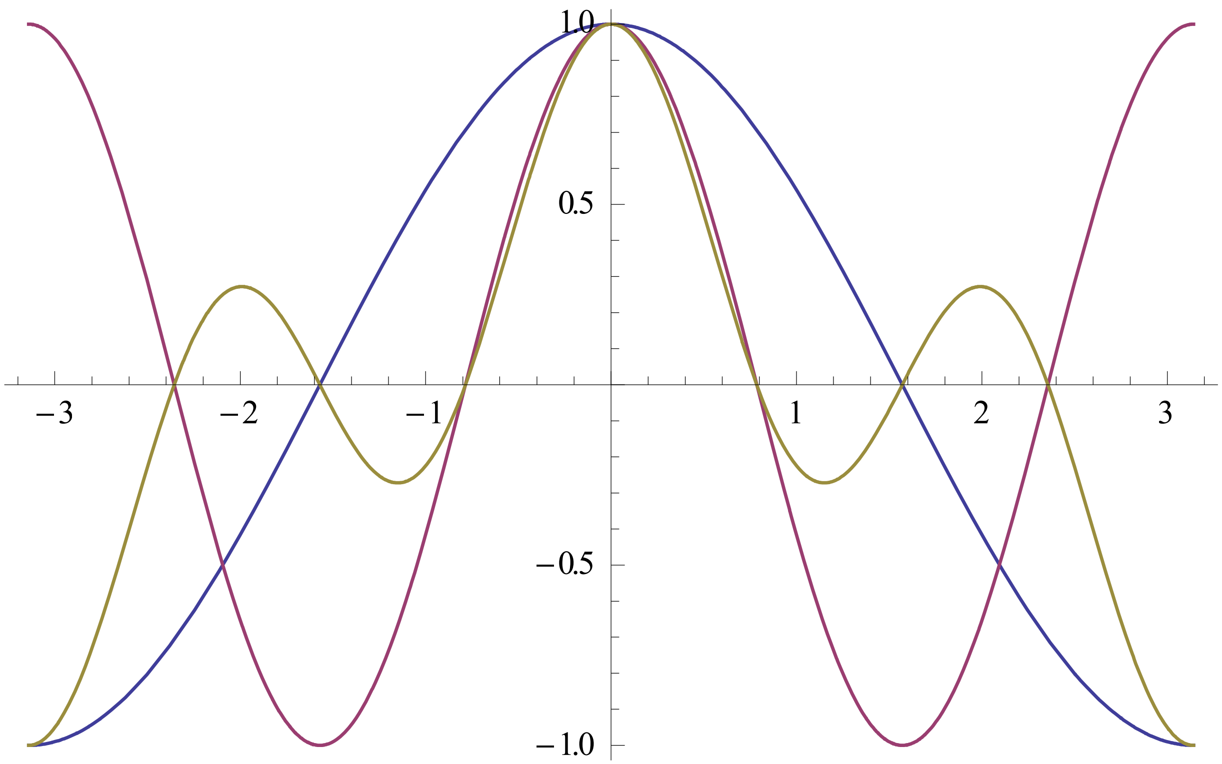
Demonstration

http://www.falstad.com/fourier/

Fourier Basis

- Why Fourier basis?
  - Can represent integrable functions with finite support (J P Fourier 1807)
- Also
  - Orthonormal in [-pi, pi]
  - Periodic signals with different frequencies
  - Continuous, differentiable basis

Orthonormality



Example: Cos(x), Cos(2x), Cos(x)*Cos(2x)

$$\int_{-pi}^{pi} \cos(x)\cos(2x)\,dx = 0$$

Fourier Transform

- Forward, mapping to frequency domain:

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt$$

- Backward, inverse mapping to time domain:

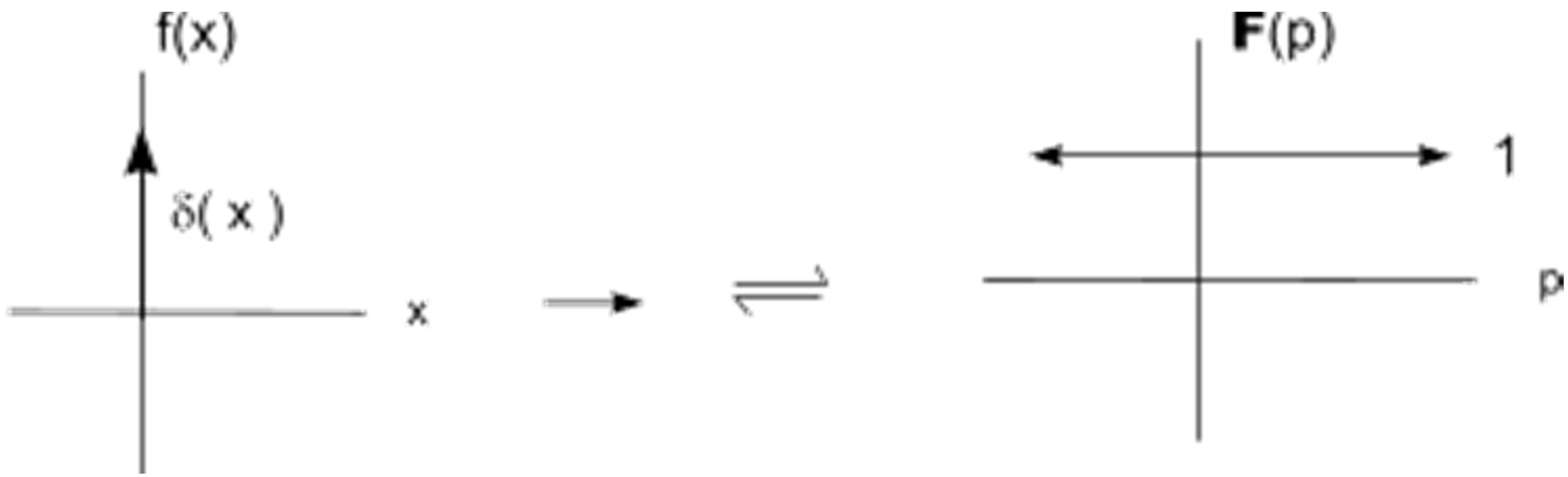
$$f(t) = \int_{-\infty}^{\infty} F(s) e^{+j2\pi st} ds$$

Sifting Property

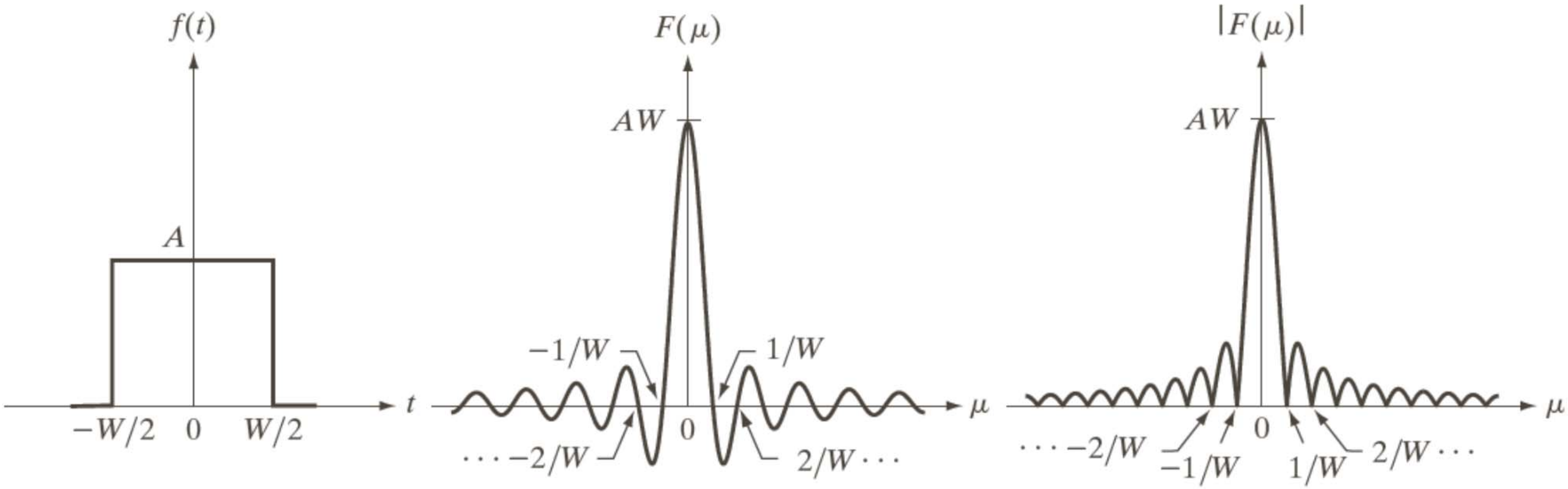
- See text book DIP 4.2.3

Common Transform Pairs

Dirac delta - constant



Chapter 4
Filtering in the Frequency Domain



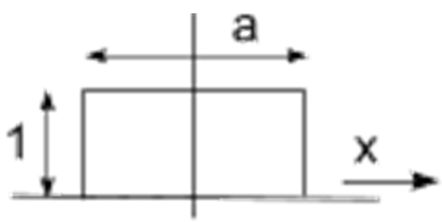
a b c

FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

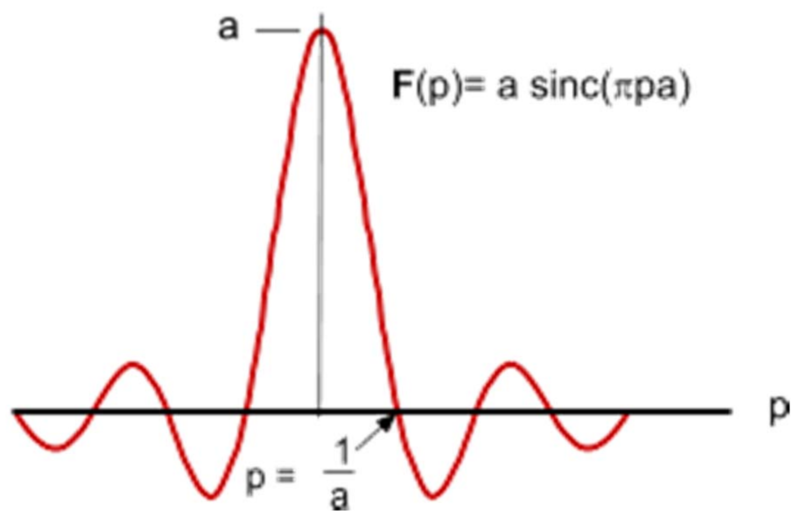
Common Transform Pairs

Rectangle – sinc

sinc(x) = sin(x) / x

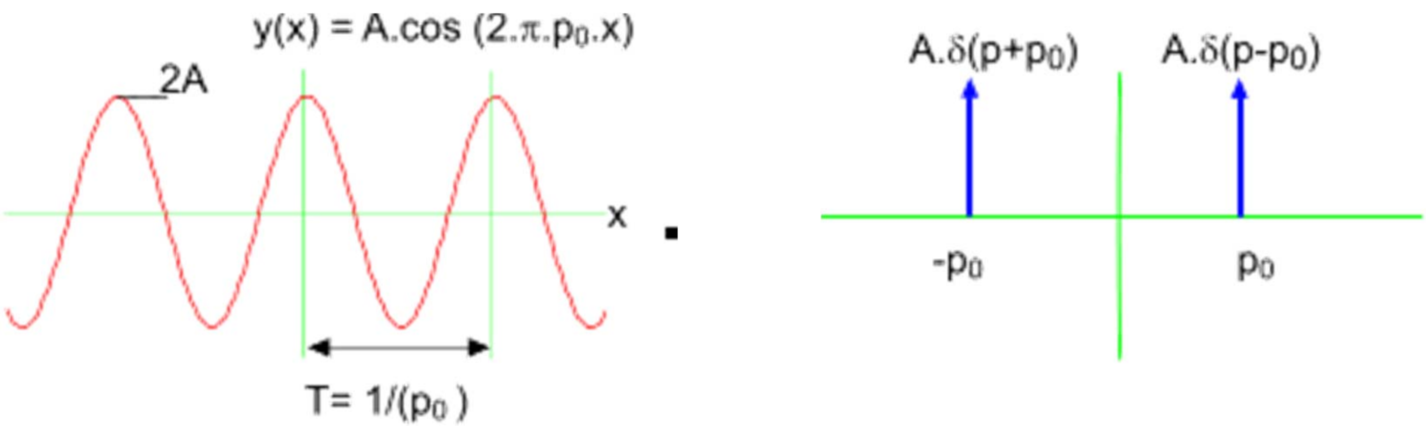


$$\Pi_a \begin{cases} = 0\ , -\infty < x < -a/2 \\ = 1\ , -a/2 < x < a/2 \\ = 0\ ,\ a/2 < x < \infty \end{cases}$$



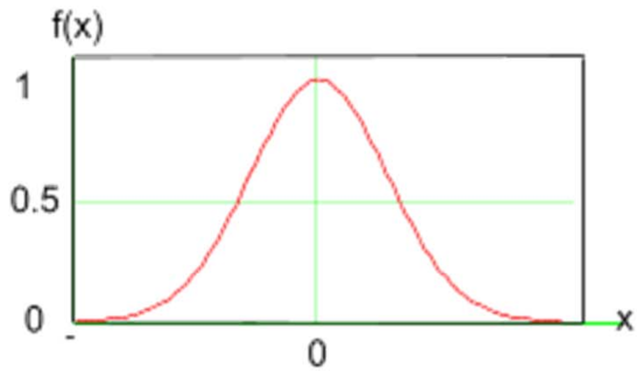
Common Transform Pairs

Cosine - Two symmetric Diracs

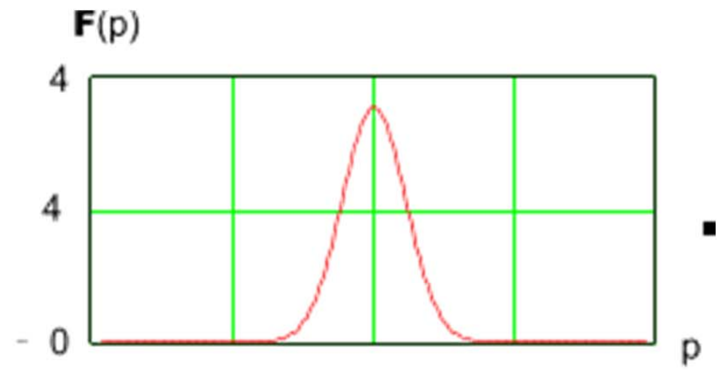


Common Transform Pairs

Gaussian – Gaussian (inverse variance)



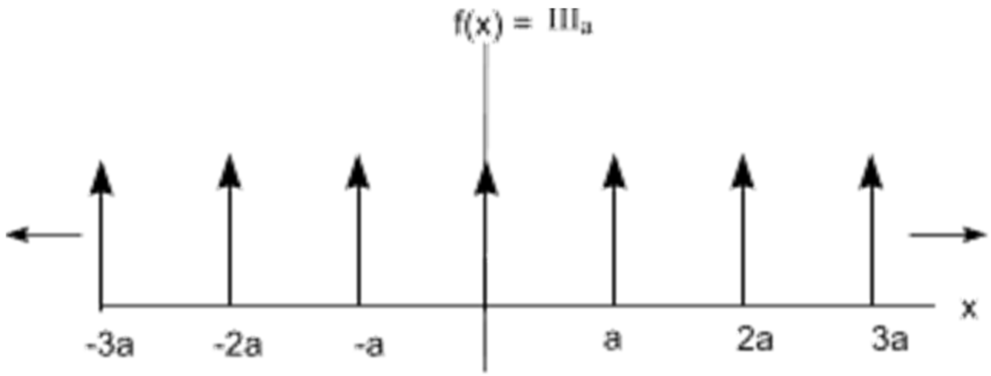
Gaussian Function

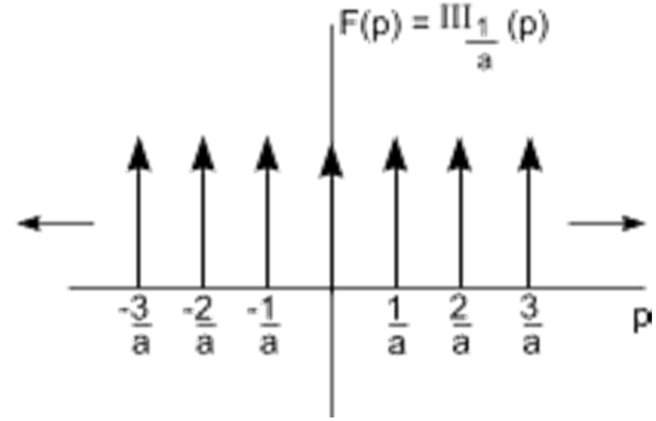


Fourier Transform

Common Transform Pairs

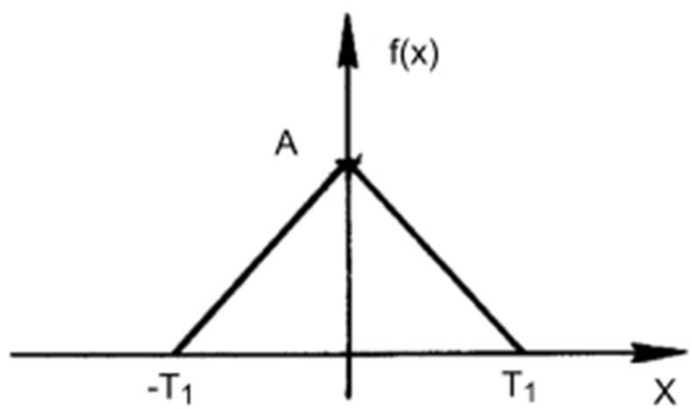
Comb – comb (inverse width)





Quiz

What is the FT of a triangle function?

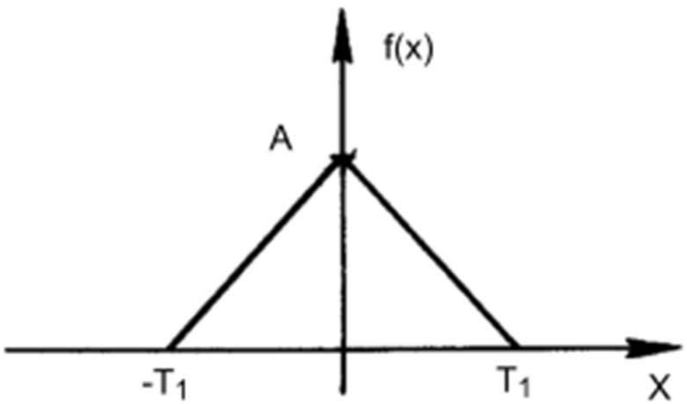


Hint: how do you get triangle function from the functions shown so far?

Answer

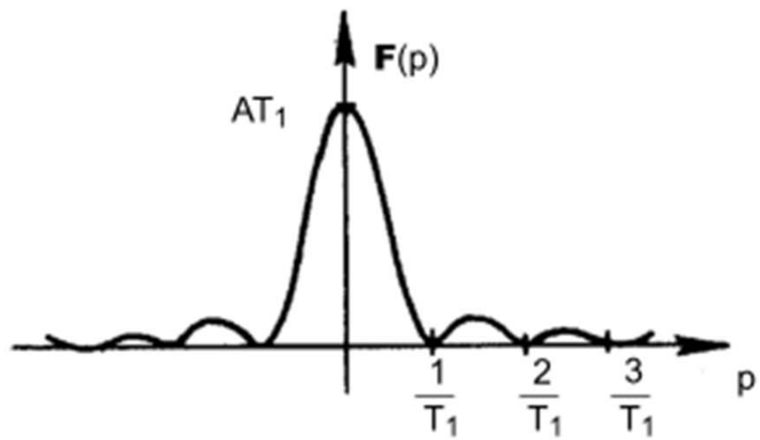
Triangle = box convolved with box

So its FT is sinc * sinc



$$f(x) = -\frac{A}{T_1}|x| + A$$

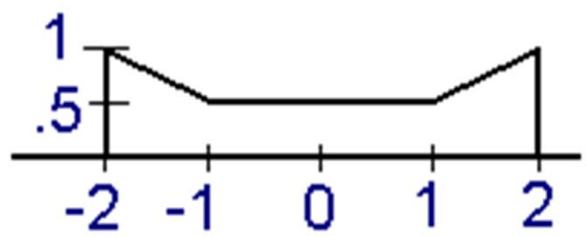
$f(x)=0 \quad |x| < T_1 \quad$ and $\quad |x| > T_1$



$$\mathbf{F}(p) = AT_1\left[\frac{\sin(\pi T_1 p)}{\pi T_1 p}\right]^2 = AT_1 \ \mathrm{sinc}^2(\pi T_1 p)$$

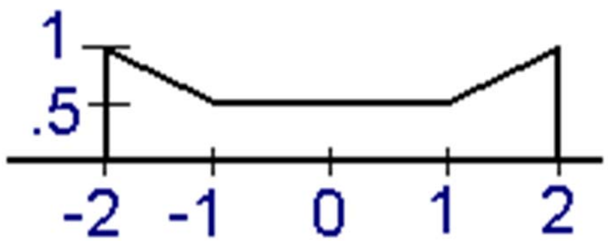
Quiz

- What is the FT?

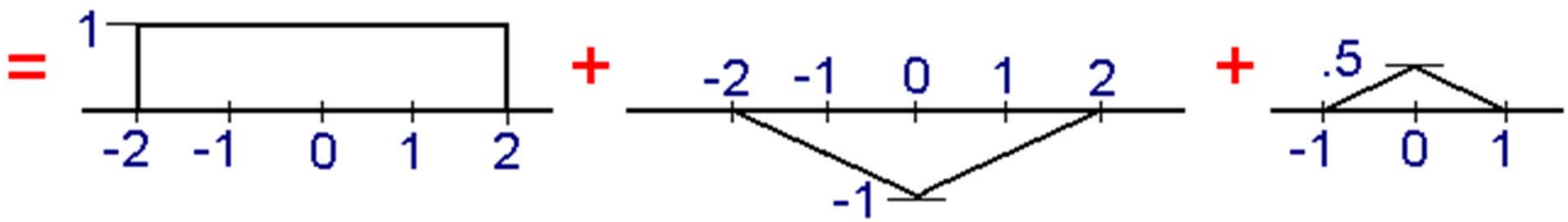


- Hint: use FT properties and express as functions with known transforms

Answer



$$f(x) = \Pi(x/4) - \Lambda(x/2) + .5\Lambda(x)$$

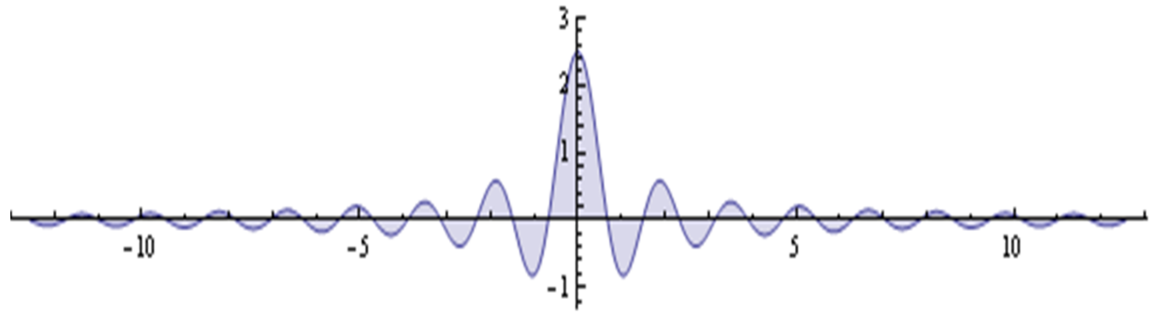


FT is linear, so

$$F(s) = 4\text{sinc}(4s) - 2\text{sinc}^2(2s) + .5\text{sinc}^2(s)$$

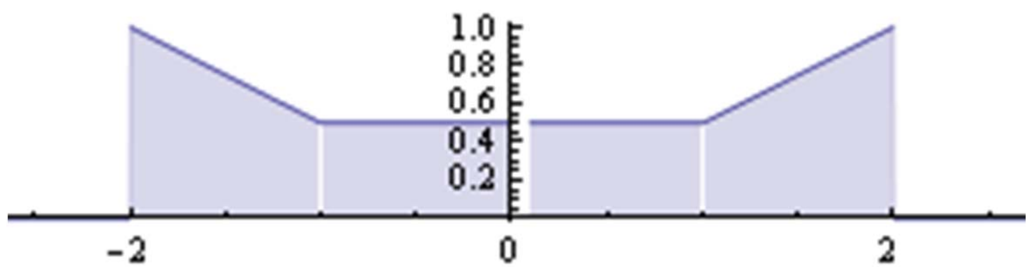
Fourier Transform

F(s) = 4sinc(4s) - 2sinc^2(2s) + .5sinc^2(s)



F(s)

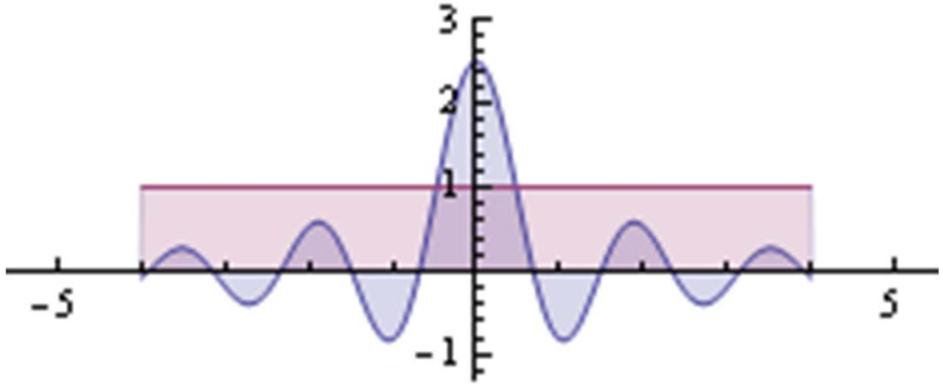
InverseFourier



f(x)

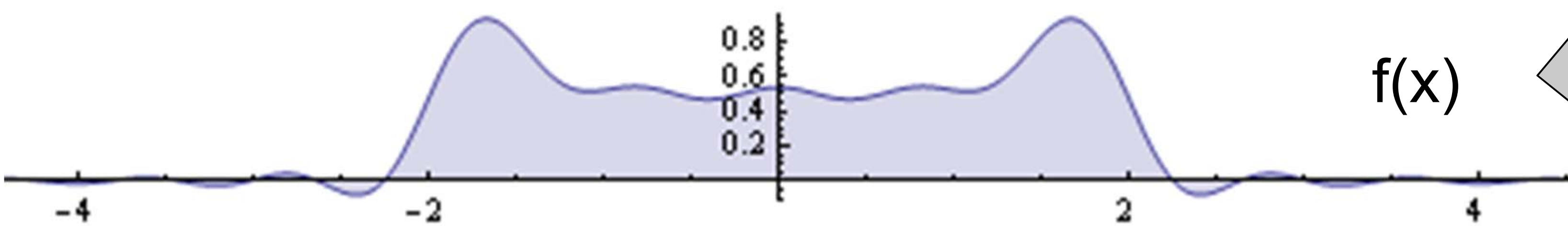
Cut-off High Frequencies

$F(s) = (4\mathrm{sinc}(4s) - 2\mathrm{sinc}^2(2s) + .5\mathrm{sinc}^2(s))*(\mathrm{HeavisidePi}(w/8)$



F(s)

InverseFourier



f(x)

1D: Common Transform Pairs Summary

Fourier Transform Pairs

Pair Number	$x(t)$	$X(f)$
1.	$\Pi\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc} \tau f$
2.	$2W \operatorname{sinc} 2Wt$	$\Pi\left(\frac{f}{2W}\right)$
3.	$\Lambda\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}^2 \tau f$
4.	$\exp(-\alpha t)u(t),\ \alpha > 0$	$\frac{1}{\alpha + j2\pi f}$
5.	$t\exp(-\alpha t)u(t),\ \alpha > 0$	$\frac{1}{(\alpha + j2\pi f)^2}$
6.	$\exp(-\alpha\|t\|),\ \alpha > 0$	$\frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
7.	$e^{-\pi(t/\tau)^2}$	$\tau e^{-\pi(f\tau)^2}$
8.	$\delta(t)$	1
9.	1	$\delta(f)$
10.	$\delta(t - t_0)$	$\exp(-j2\pi f t_0)$
11.	$\exp(j2\pi f_0 t)$	$\delta(f - f_0)$
12.	$\cos 2\pi f_0 t$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$
13.	$\sin 2\pi f_0 t$	$\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$
14.	$u(t)$	$(j2\pi f)^{-1} + \frac{1}{2}\delta(f)$
15.	$\operatorname{sgn} t$	$(j\pi f)^{-1}$
16.	$\frac{1}{\pi t}$	$-j\operatorname{sgn}(f)$
17.	$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\lambda)}{t - \lambda}\,d\lambda$	$-j\operatorname{sgn}(f)X(f)$
18.	$\sum_{m=-\infty}^{\infty}\delta(t - mT_s)$	$f_s \sum_{m=-\infty}^{\infty}\delta(f - mf_s),\ f_s = T_s^{-1}$

source

FT Properties: Convolution

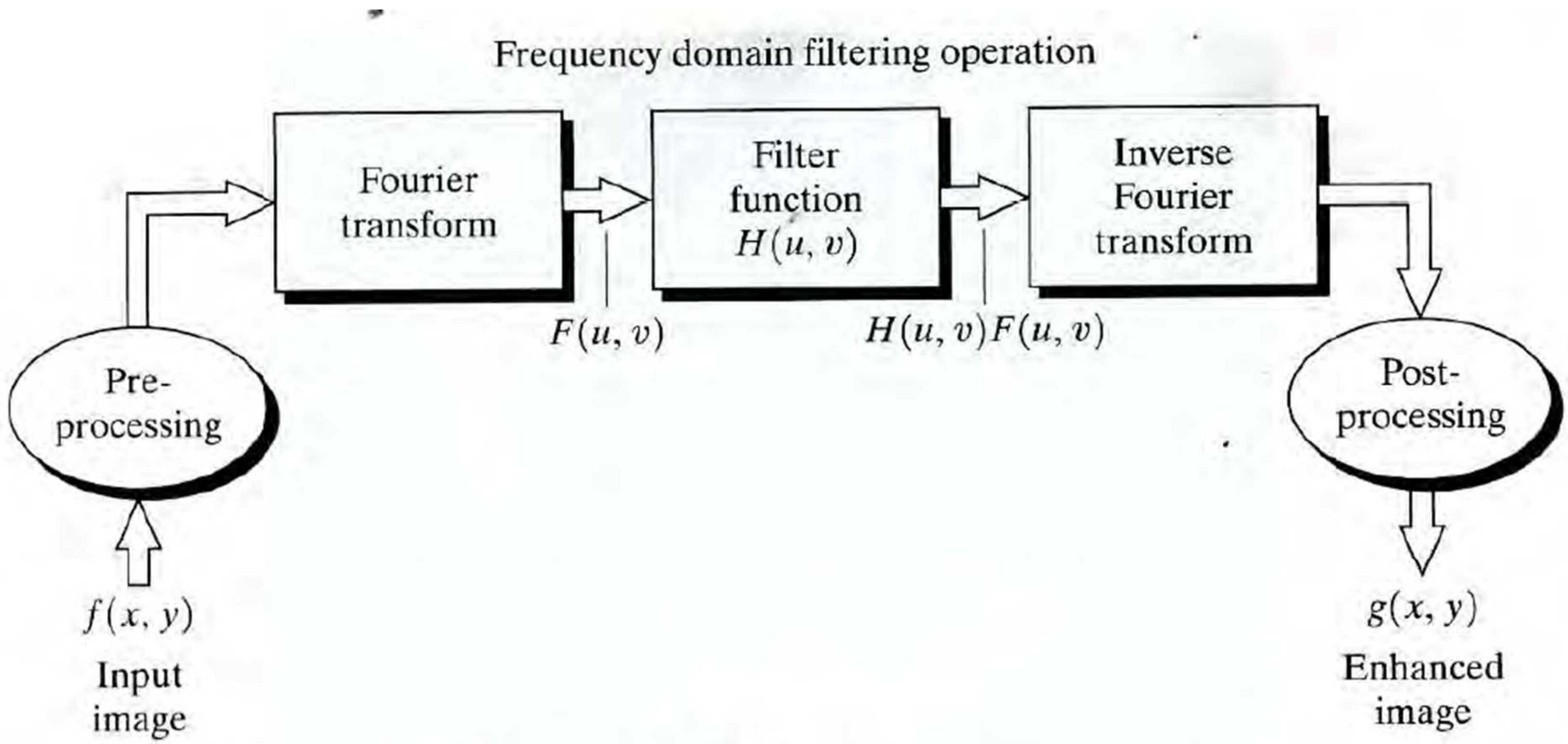
- See book DIP 4.2.5:

$$\mathcal{F}[f(t) \otimes g(t)] = F(s).G(s)$$

- Convolution in space/time domain is equiv. to multiplication in frequency domain.

Time Convolution	$f(t) \star g(t)$	$\leftrightarrow$	$F(\omega)G(\omega)$
Frequency Convolution	$f(t)g(t)$	$\leftrightarrow$	$\frac{1}{2\pi}F(\omega) \star G(\omega)$

Important Application



Filtering in frequency Domain

FT Properties

Linearity	$\alpha f(t) + \beta g(t)$	$\leftrightarrow$	$\alpha F(\omega) + \beta G(\omega)$
Time Translation	$f(t - t_0)$	$\leftrightarrow$	$e^{-j\omega t_0} F(\omega)$
Scale Change	$f(at)$	$\leftrightarrow$	$\frac{1}{\lVert a \rVert} F(\omega / a)$
Frequency Translation	$e^{j\omega_0 t} f(t)$	$\leftrightarrow$	$F(\omega - \omega_0)$
Time Convolution	$f(t) \star g(t)$	$\leftrightarrow$	$F(\omega) G(\omega)$
Frequency Convolution	$f(t) g(t)$	$\leftrightarrow$	$\frac{1}{2\pi} F(\omega) \star G(\omega)$

$$(f * g)(x) = \int_{\mathbf{R}^d} f(y) g(x - y)\, dy = \int_{\mathbf{R}^d} f(x - y) g(y)\, dy.$$