

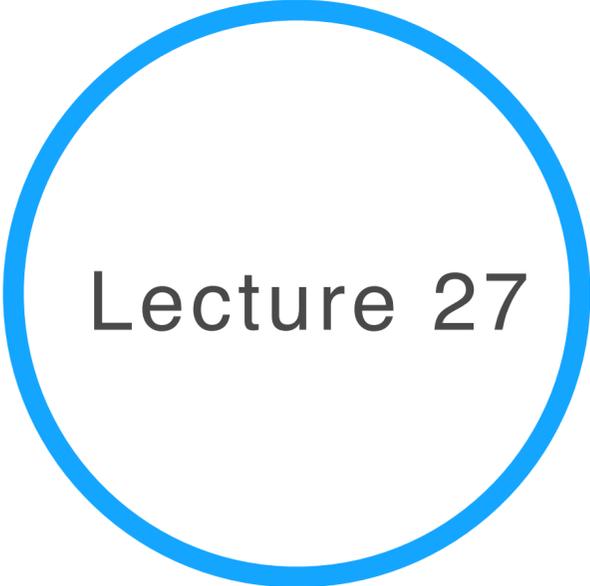
Advanced Data Visualization

CS 6965

Fall 2019

Prof. Bei Wang Phillips

University of Utah



Lecture 27

An Introduction to Discrete Morse Theory

TOPO

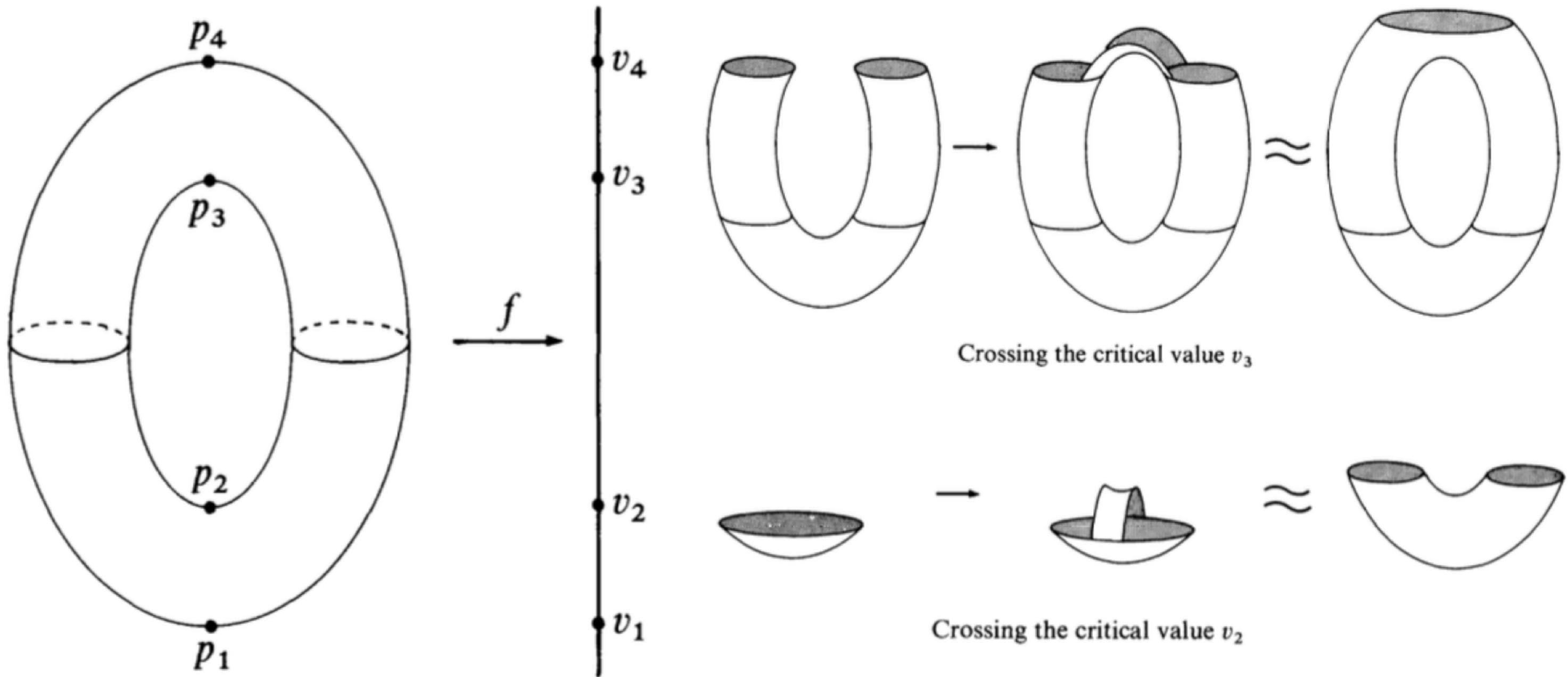
About...

- This lecture can be mathematical...the goal is to give a brief introduction to Discrete Morse Theory and its applications

Classic Morse Theory

(CMT)

CMT Example

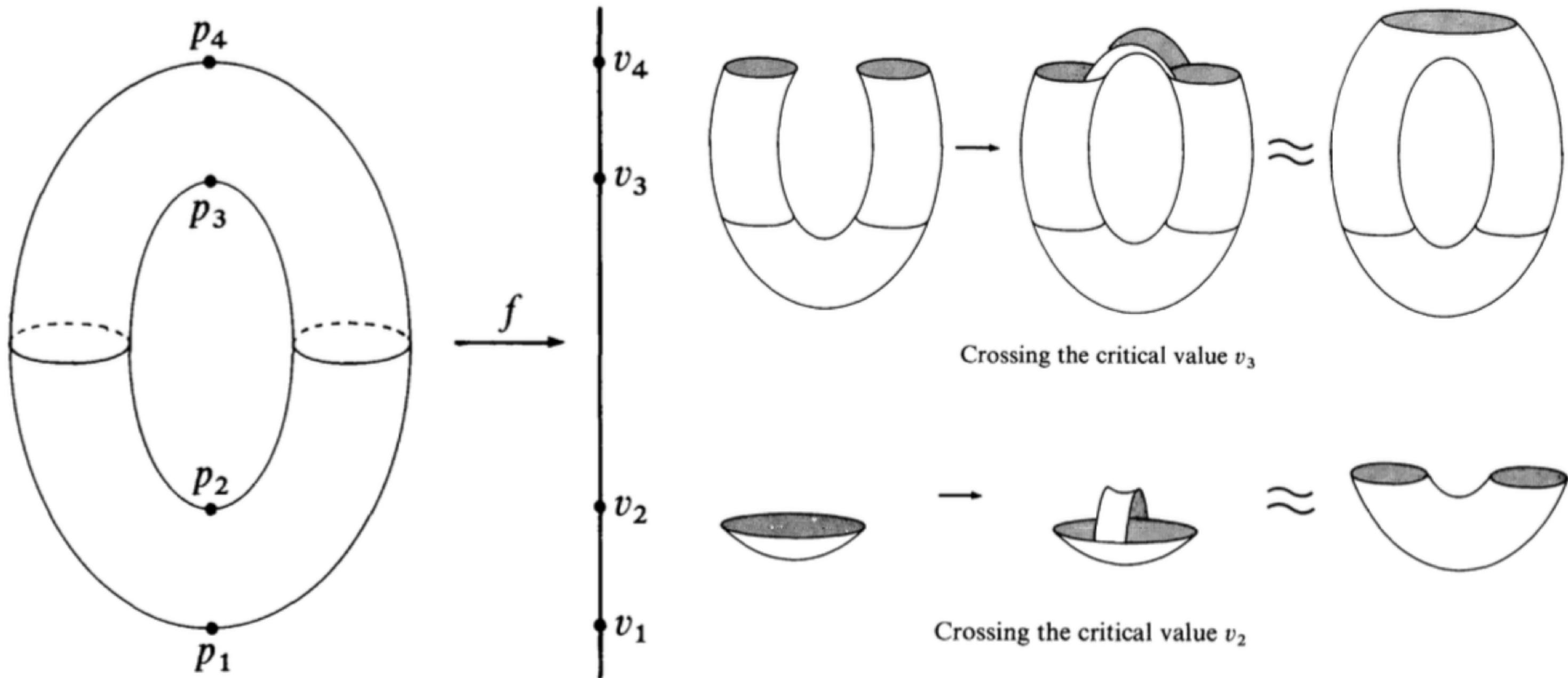


[GoreskyMacPherson1988]

Definitions

CMT studies the topological change of \mathbb{X}_a as a varies.

- \mathbb{X} : a compact, smooth d -manifold
- $f : \mathbb{X} \rightarrow \mathbb{R}$: differentiable
- *sublevel set*: $\mathbb{X}_a = f^{-1}(-\infty, a]$
- A point $x \in \mathbb{X}$ is *critical* if the derivative at x equals zero
- $\lambda(x)$: the *Morse index* of a non-degenerate critical point x is the number of negative eigenvalues in the Hessian matrix
- Next page: p_1, p_2, p_3, p_4 , index 0, 1, 1, and 2
- f is a *Morse function* if all critical points are non-degenerate and its values at the critical points are distinct



[GoreskyMacPherson1988]

Fundamental Results of CMT

Theorem (CMT-A)

Let $f : \mathbb{X} \rightarrow \mathbb{R}$ be a differentiable function on a compact smooth manifold \mathbb{X} .

Let $a < b$ be real values such that $f^{-1}[a, b]$ is compact and contains no critical points of f .

Then \mathbb{X}_a is diffeomorphic to \mathbb{X}_b .

Fundamental Results of CMT

Theorem (CMT-B)

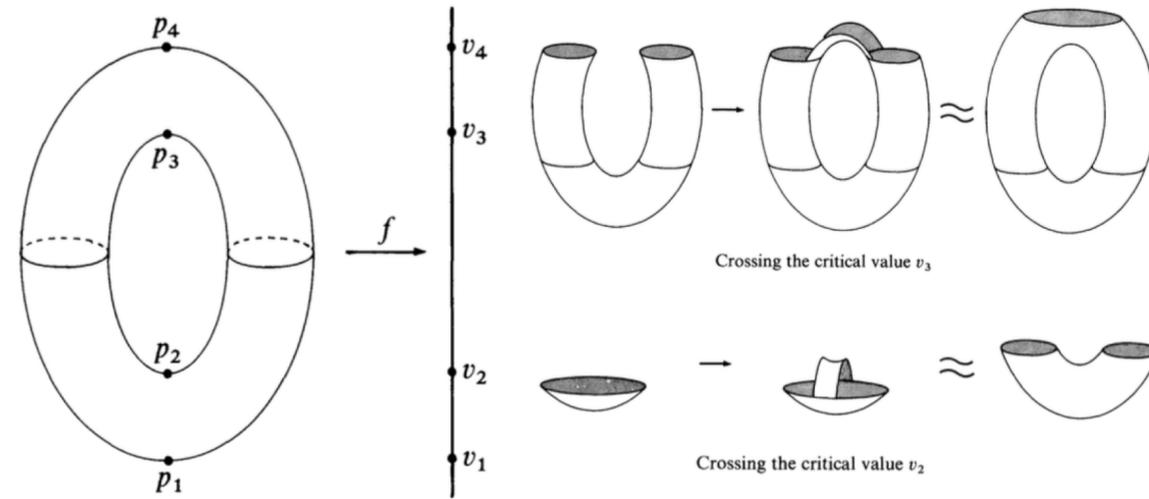
Let f be a Morse function on \mathbb{X} .

Consider two regular values $a < b$ such that $f^{-1}[a, b]$ is compact but contains one critical point u of f , with index λ .

Then \mathbb{X}_b is homotopy equivalent (diffeomorphic) to the space $\mathbb{X}_a \cup_B A$, that is, by attaching A along B .

The pair of spaces $(A, B) = (D^\lambda \times D^{d-\lambda}, (\partial D^\lambda) \times D^{d-\lambda})$ is the Morse data, where d is the dimension of \mathbb{X} and λ is the Morse index of u , D^k denotes the closed k -dimensional disk and ∂D^k is its boundary.

$$(A, B) = (D^\lambda \times D^{d-\lambda}, (\partial D^\lambda) \times D^{d-\lambda})$$



[GoreskyMacPherson1988]

Critical point

Morse data (A, B)

p_1

$$\left(\text{shaded circle}, \emptyset \right) = (D^0 \times D^2, \partial D^0 \times D^2)$$

p_2 or p_3

$$\left(\text{shaded rectangle}, | \quad | \right) = (D^1 \times D^1, \partial D^1 \times D^1)$$

p_4

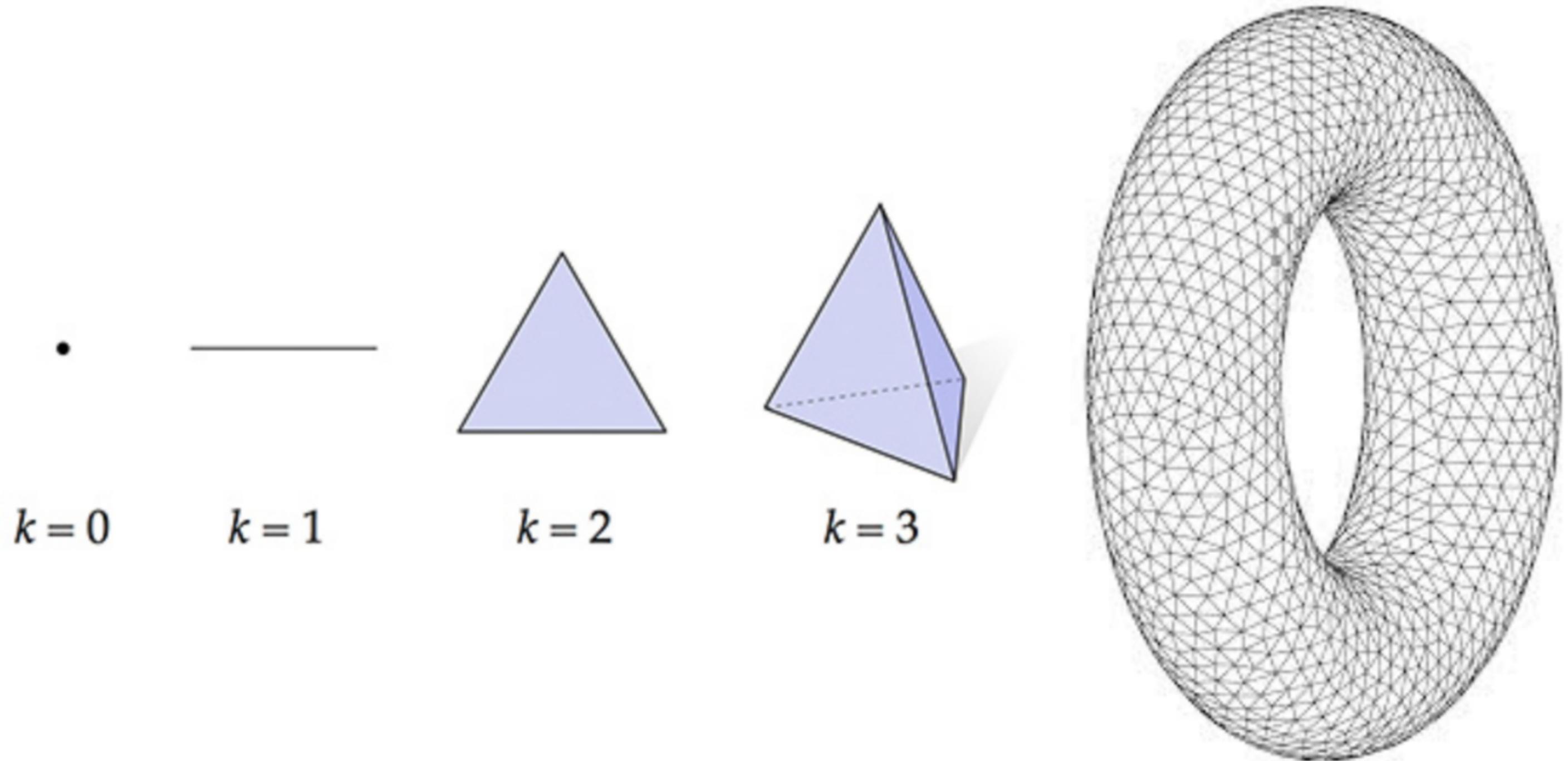
$$\left(\text{shaded circle}, \text{circle} \right) = (D^2 \times D^0, \partial D^2 \times D^0)$$

[GoreskyMacPherson1988]

Discrete Morse Theory

(DMT)

Data as a simplicial complex



DMT Basic Setup

DMT is the combinatorial version of CMT.

- K : a finite simplicial complex
- $\alpha^{(p)} \in K$: a simplex of dimension p .
- $\alpha < \beta$: α is a face of simplex β .
- $U(\alpha) = \{\beta^{(p+1)} > \alpha \mid f(\beta) \leq f(\alpha)\}$
- $L(\alpha) = \{\gamma^{(p-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\}$
- $U(\alpha)$ contains the higher-dimensional cofaces of α with lower (or equal) function values
- $L(\alpha)$ contains the lower-dimensional faces of α with higher (or equal) function values.
- Let $|U(\alpha)|$ and $|L(\alpha)|$ be their sizes.

Discrete Morse Function

Definition

A function $f : K \rightarrow \mathbb{R}$ is a *discrete Morse function* if for every $\alpha^{(p)} \in K$,

- (i) $|U(\alpha)| \leq 1$ and
- (ii) $|L(\alpha)| \leq 1$.

Definition

A simplex $\alpha^{(p)}$ is *critical* if (i) $|U(\alpha)| = 0$ and (ii) $|L(\alpha)| = 0$. A *critical value* of f is its value at a critical simplex.

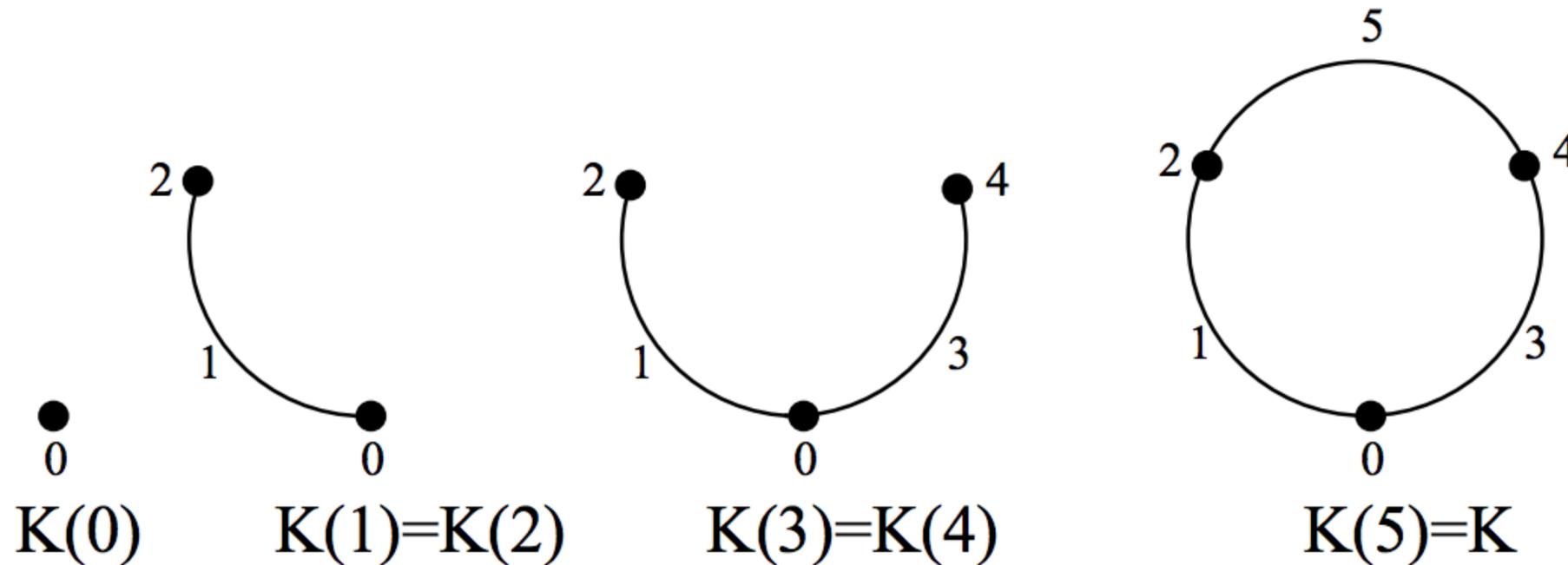
Discrete Morse Function

Definition

A simplex $\alpha^{(p)}$ is *noncritical* if either of the following conditions holds: (i) $|U(\alpha)| = 1$; (ii) $|L(\alpha)| = 1$; as noted above these conditions can not both be true ([Forman1998], Lemma 2.5).

Level Subcomplex

- Given $c \in \mathbb{R}$, we have the *level subcomplex*
 $K_c = \cup_{f(\alpha) \leq c} \cup_{\beta \leq \alpha} \beta$.
- K_c contains all simplices α of K such that $f(\alpha) \leq c$ along with all of their faces.



The level subcomplexes of the discrete Morse function shown in Figure 2.2(ii)

Fundamental Results of DMT

Theorem (DMT-A)

Suppose the interval $[a, b]$ contains no critical value of f . Then K_b is homotopy equivalent to K_a . In fact, K_a is a deformation retract of K_b and moreover, K_b simplicially collapses onto K_a .

Fundamental Results of DMT

Theorem (DMT-A)

Suppose the interval $[a, b]$ contains no critical value of f . Then K_b is homotopy equivalent to K_a . In fact, K_a is a deformation retract of K_b and moreover, K_b simplicially collapses onto K_a .

Fundamental Results of DMT

Theorem (DMT-B)

Suppose $\sigma^{(p)}$ is a critical simplex with $f(\sigma) \in (a, b]$, and there are no other critical simplices with values in $(a, b]$. Then $K(b)$ is homotopy equivalent to attaching a p -cell $e^{(p)}$ along its entire boundary; that is, $K_b = K_a \cup_{e^{(p)}} e^{(p)}$.

Discrete Gradient Vector Fields

- Given a discrete Morse function $f : K \rightarrow \mathbb{R}$ we may associate a discrete gradient vector field.
- Since any noncritical simplex $\alpha^{(p)}$ has at most one of the sets $U(\alpha)$ and $L(\alpha)$ nonempty, there is a unique face $\nu^{(p-1)} < \alpha$ with $f(\nu) \geq f(\alpha)$ or a unique coface $\beta^{(p+1)} > \alpha$ with $f(\beta) \leq f(\alpha)$.
- Denote by V the collection of all such pairs $\{\sigma < \tau\}$.
- Then every simplex in K is in at most one pair in V and the simplices not in any pair are precisely the critical cells of the function f .
- We call V the *gradient vector field associated to f* .
- We visualize V by drawing an arrow from α to β for every pair $\{\alpha < \beta\} \in V$.

Discrete Gradient Vector Fields

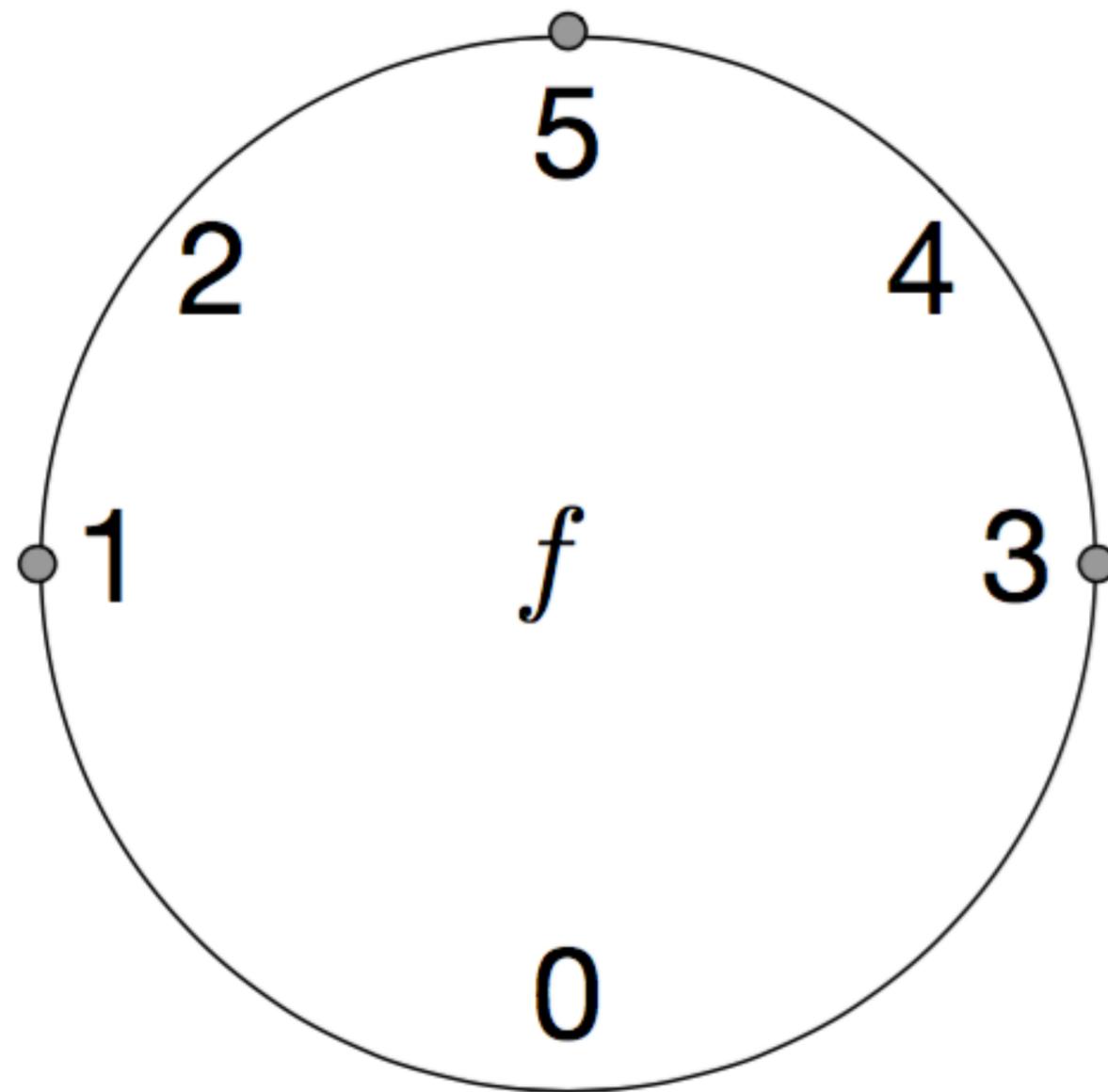
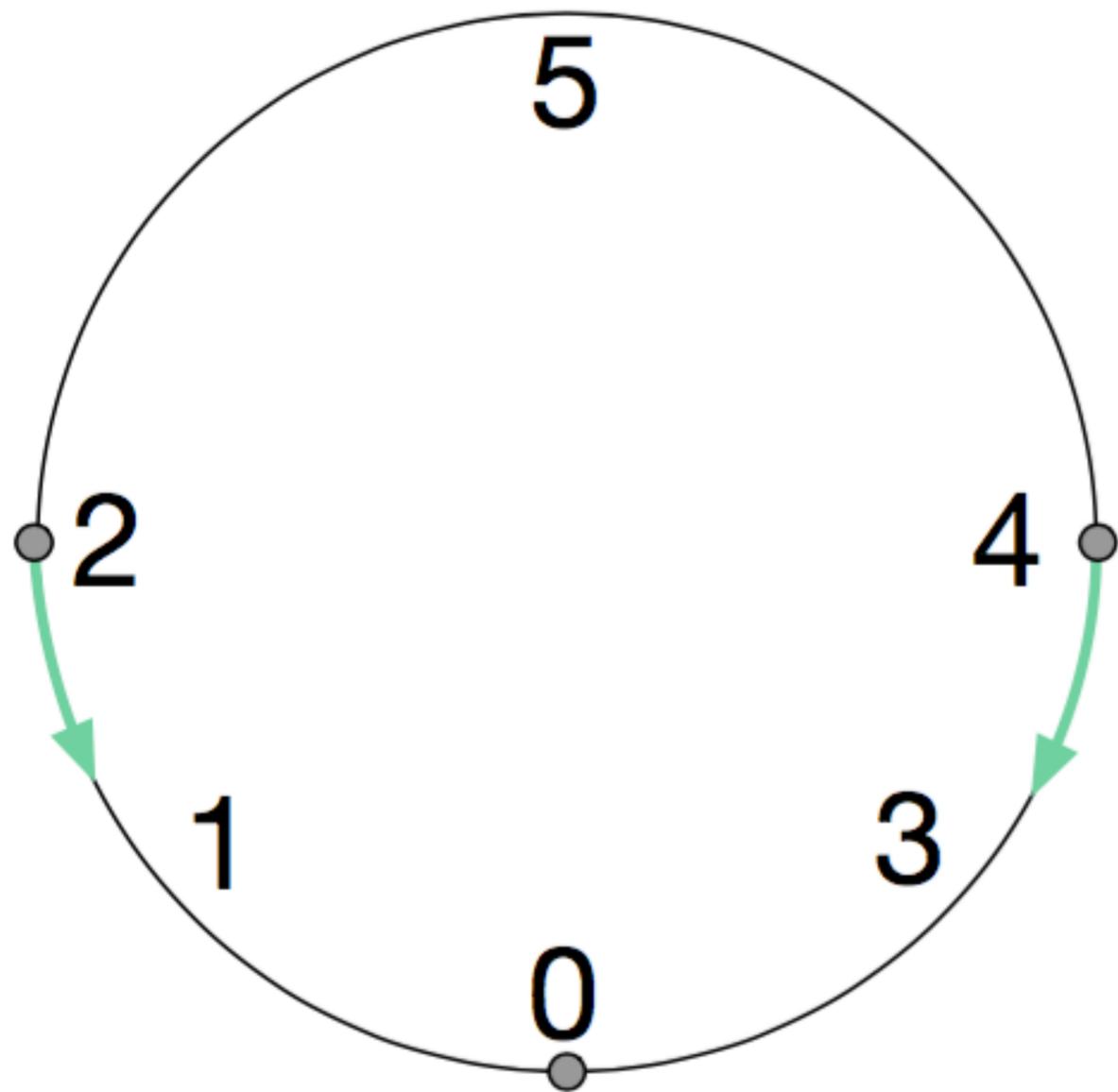
- Data reduction: collapsing the pairs in V using the arrows.
- By a V -path, we mean a sequence

$$\alpha_0^{(p)} < \beta_0^{(p+1)} > \alpha_1^{(p)} < \beta_1^{(p+1)} > \dots < \beta_r^{(p+1)} > \alpha_{r+1}^{(p)}$$

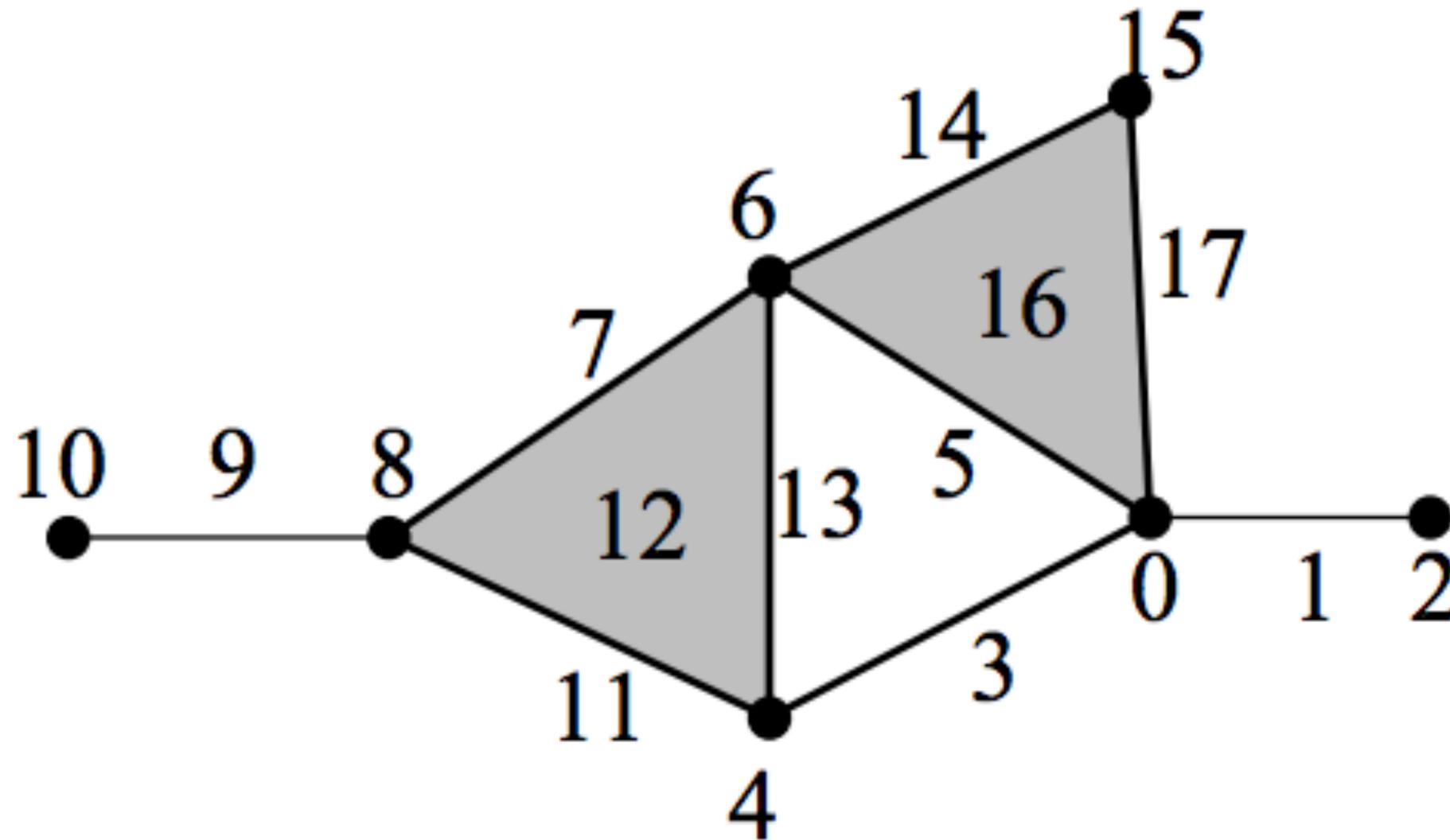
where each $\{\alpha_i < \beta_i\}$ is a pair in V . Such a path is *nontrivial* if $r > 0$ and *closed* if $\alpha_{r+1} = \alpha_0$.

Theorem ([Forman1998])

If V is a gradient vector field associated to a discrete Morse function f on K , then V has no nontrivial closed V -paths.

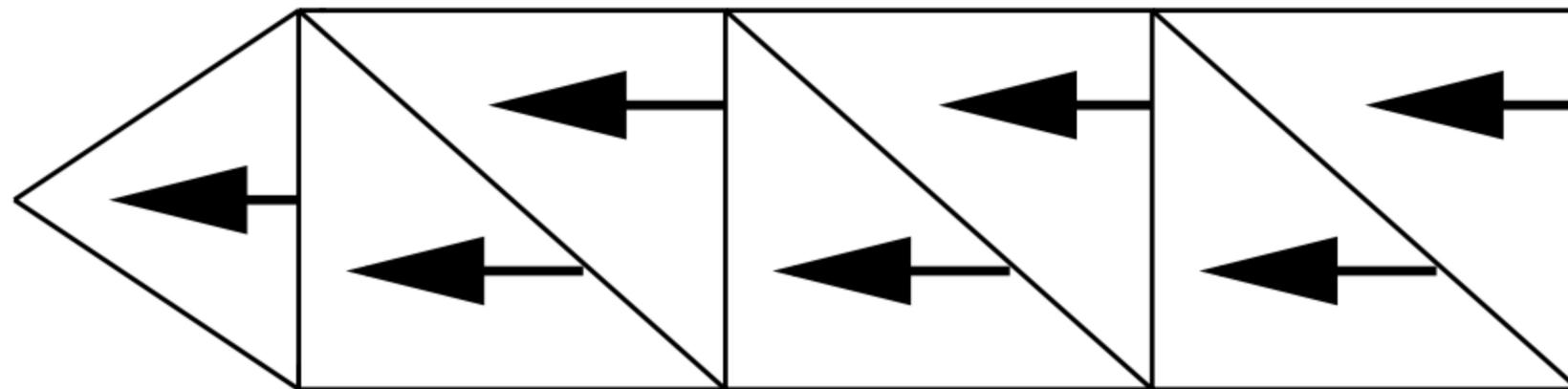
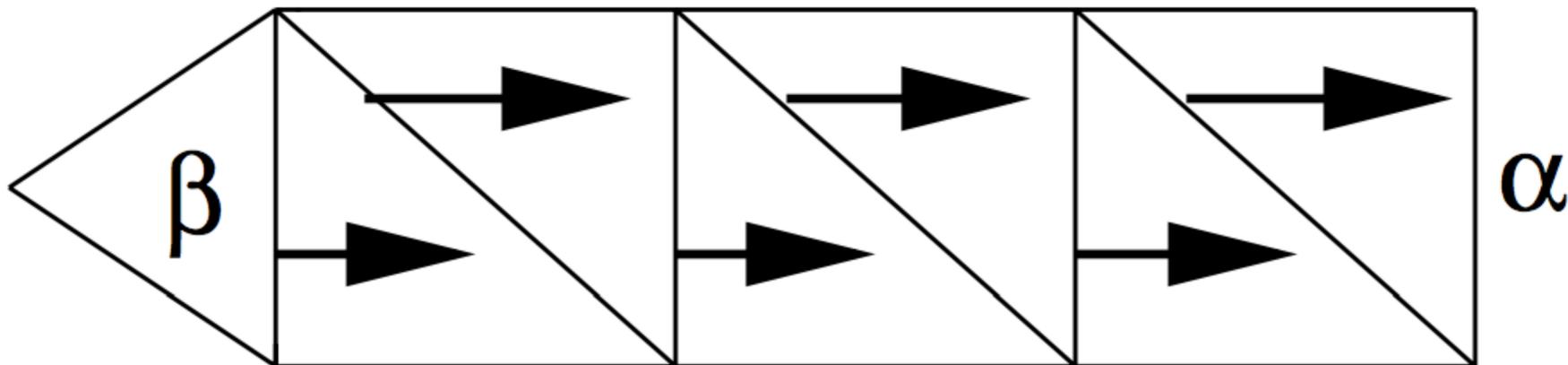


Draw the DVF



DMT

Examples



Cancelling critical points.

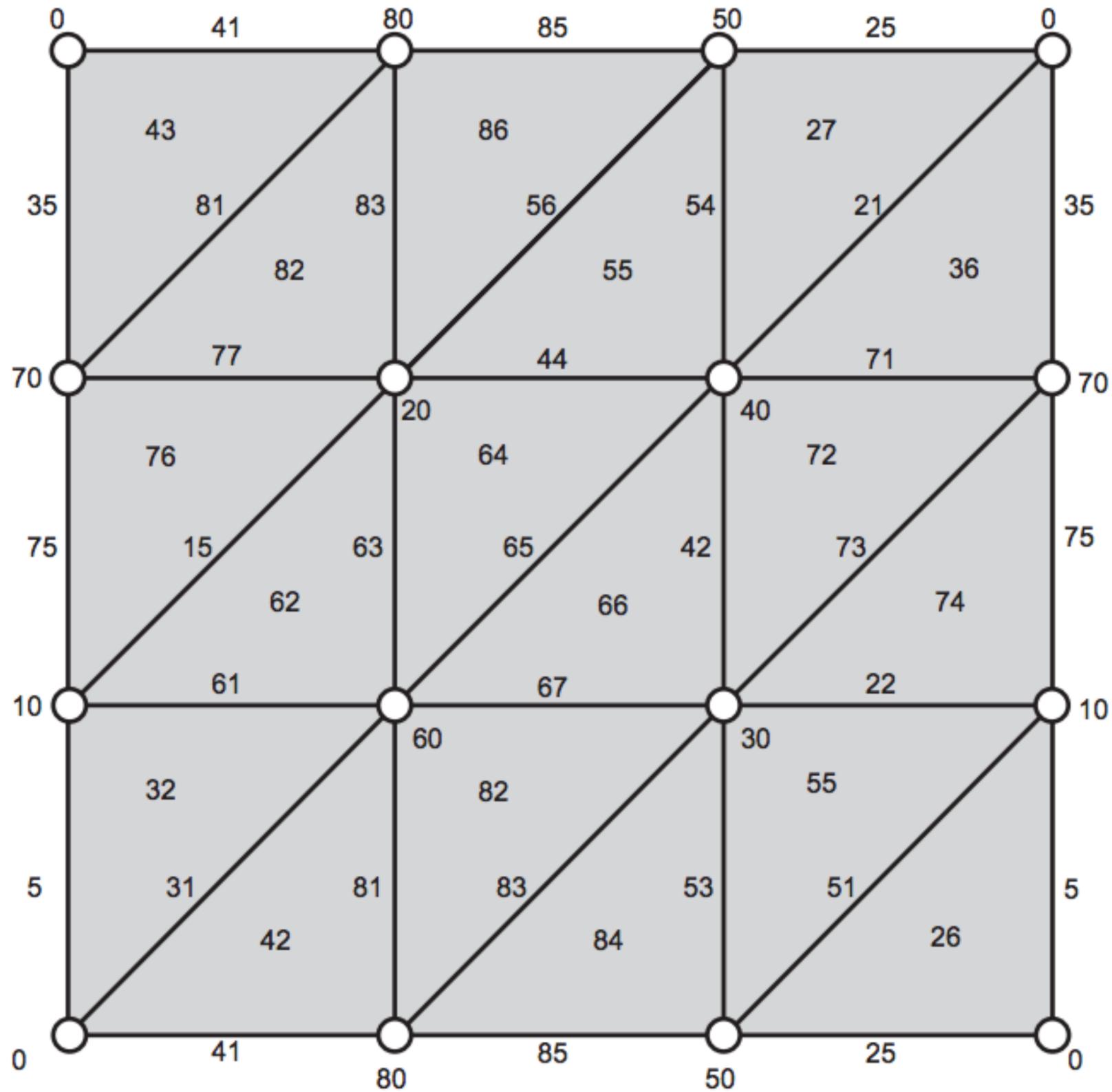


FIGURE 1. A discrete Morse function on the torus.

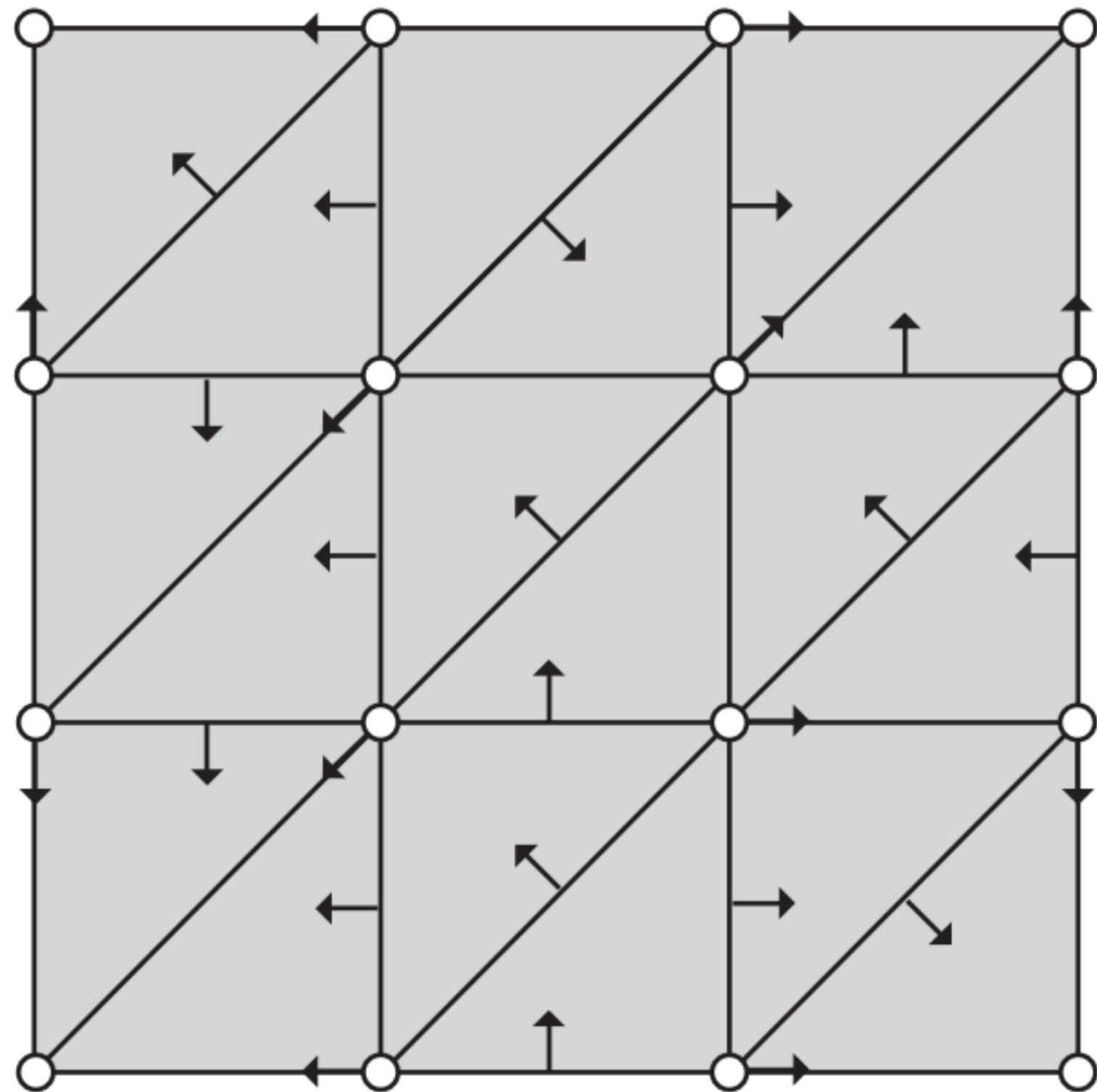
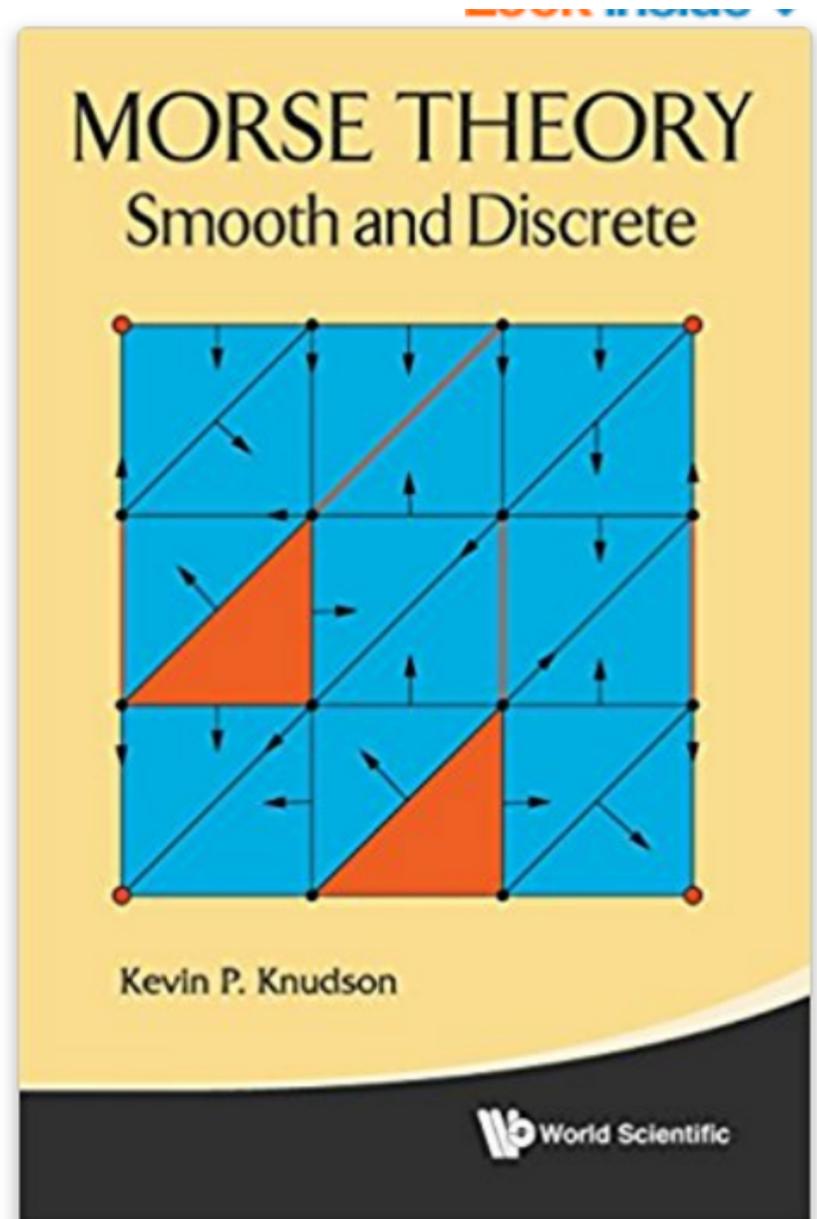
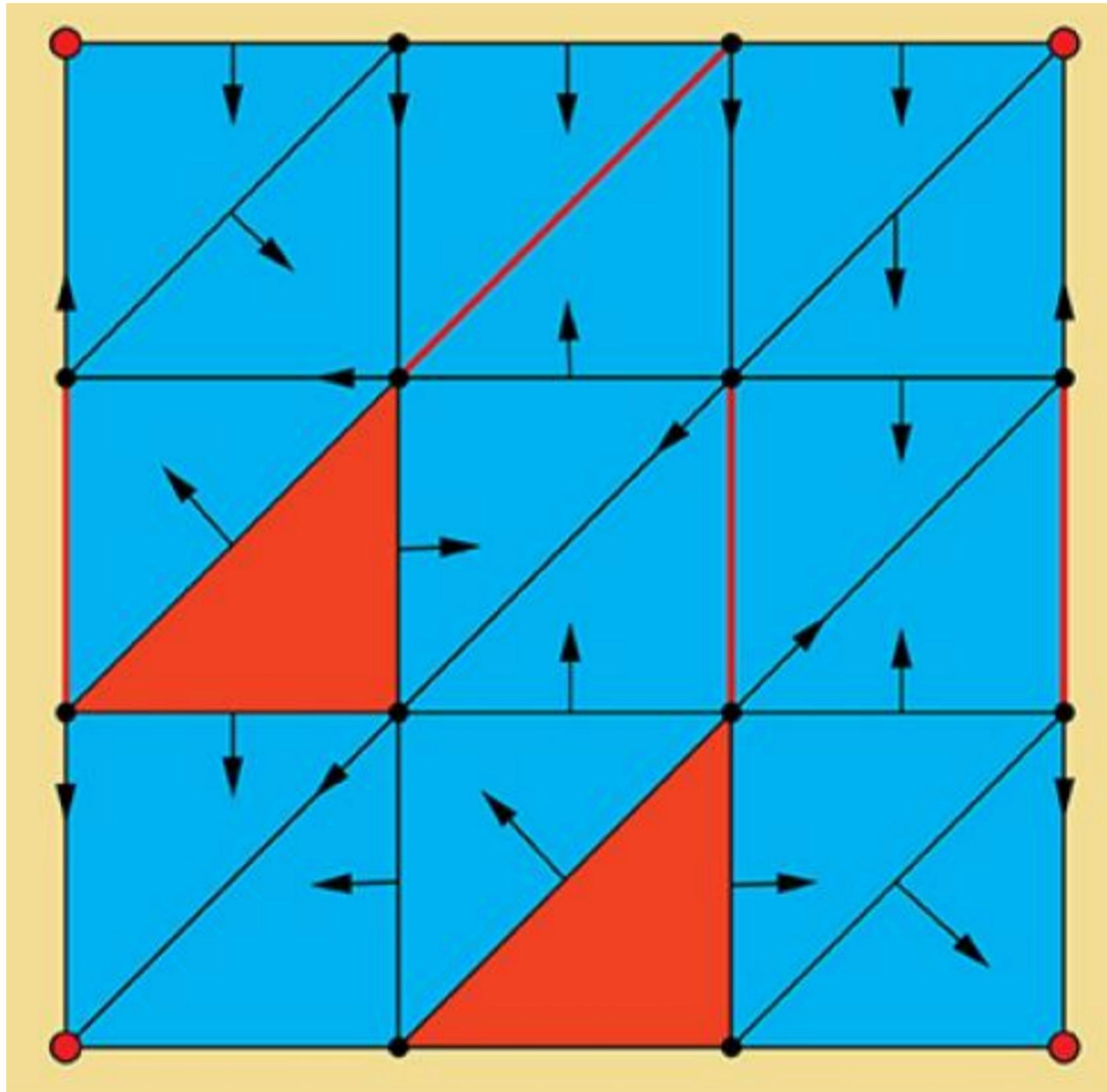


FIGURE 2. A gradient vector field on the torus.



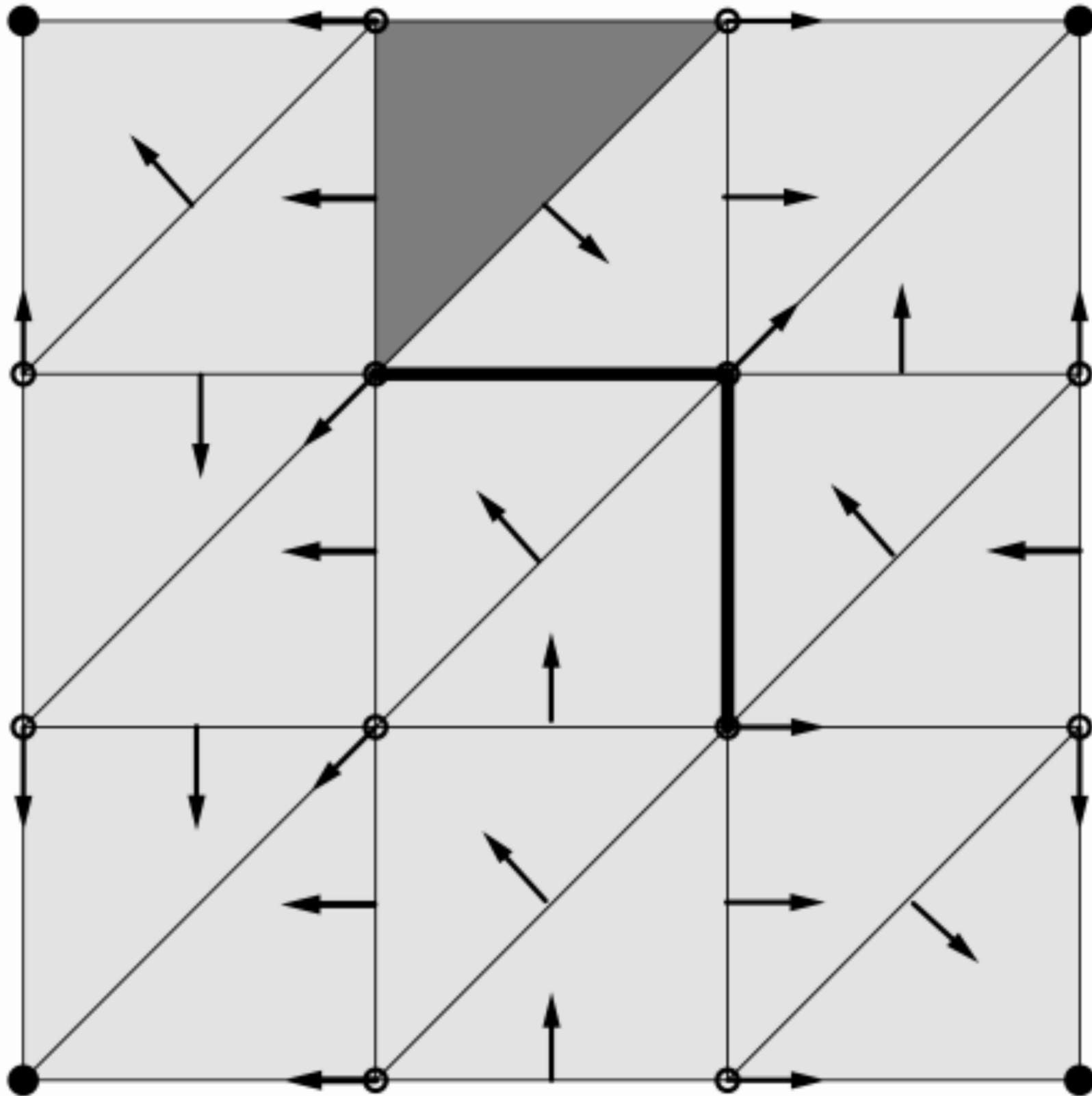


FIGURE 3. A discrete vector field on a triangulated torus. There are four critical cells: the top-center triangle (dark gray), the top and right edges of the center square (indicated with thicker lines), and the vertex obtained by identifying the four corners.

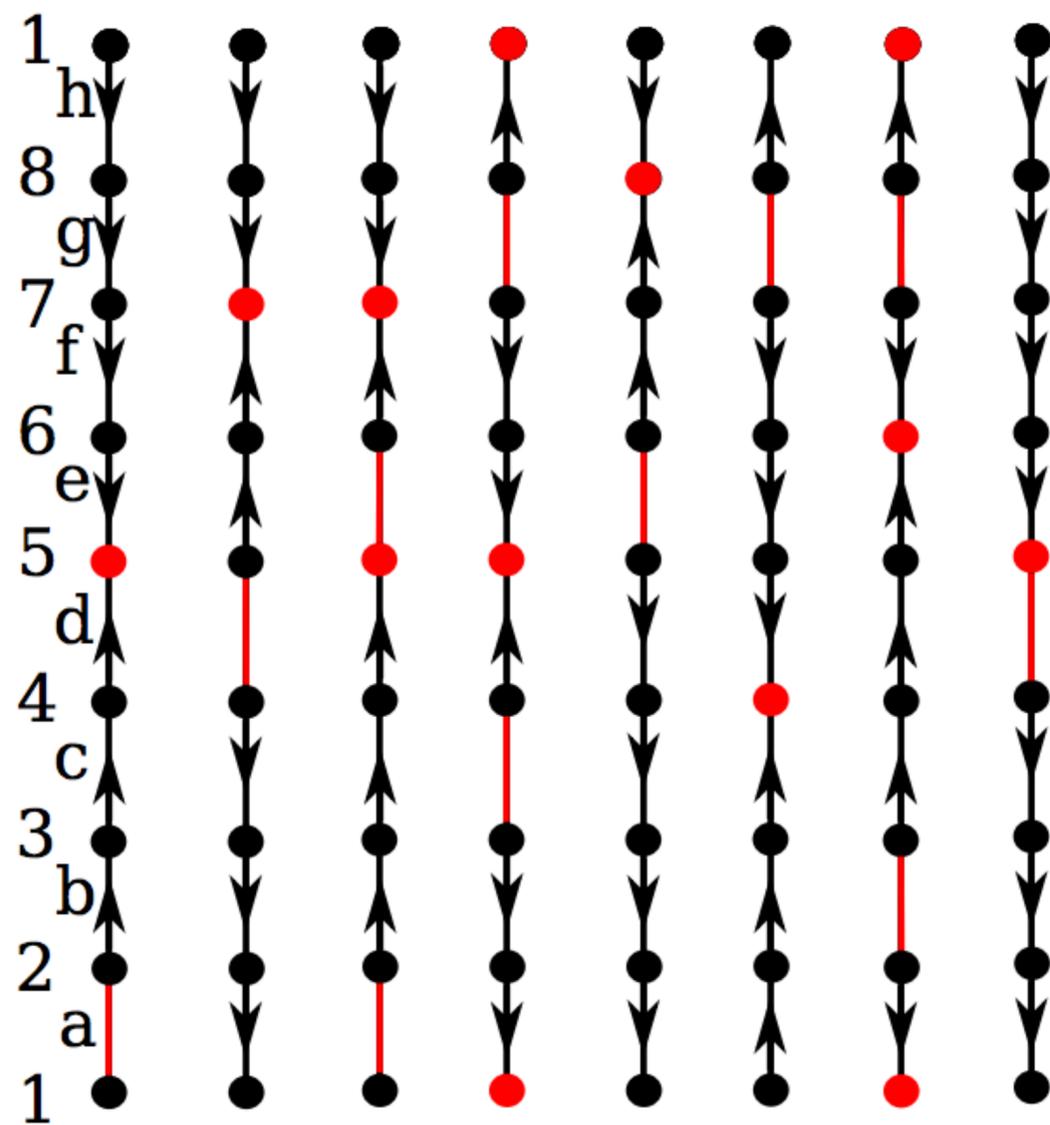


FIGURE 4. A family of discrete gradient fields on the circle. The slices are numbered left to right as $0, 1, \dots, 7$.

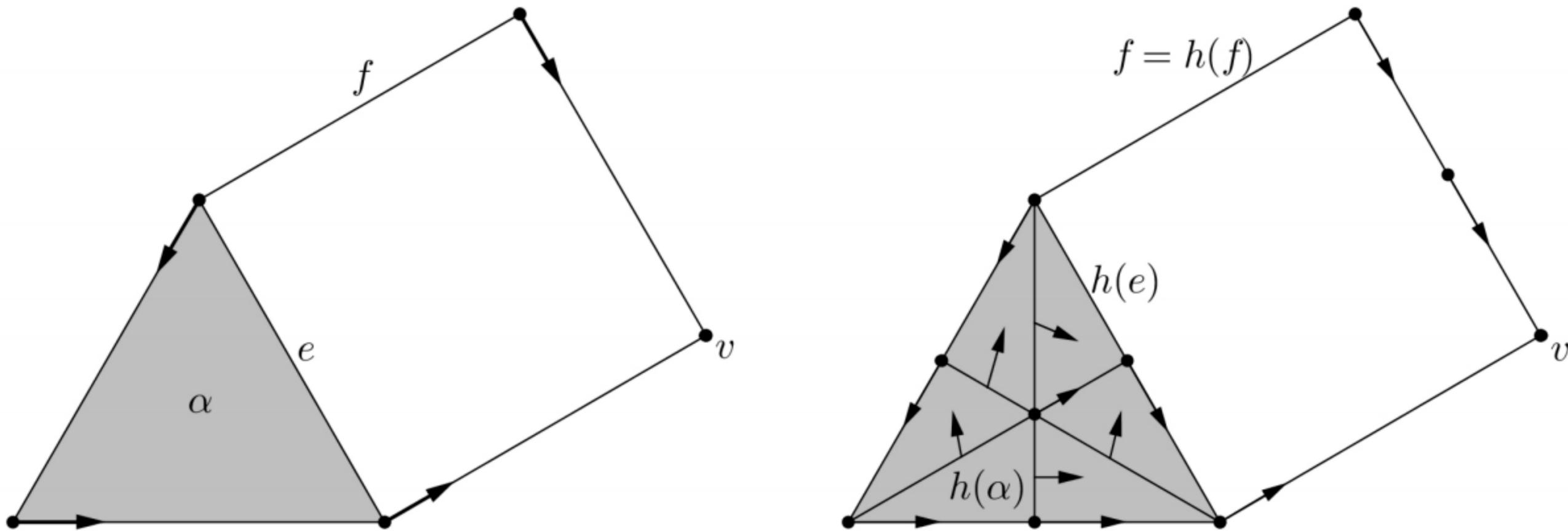
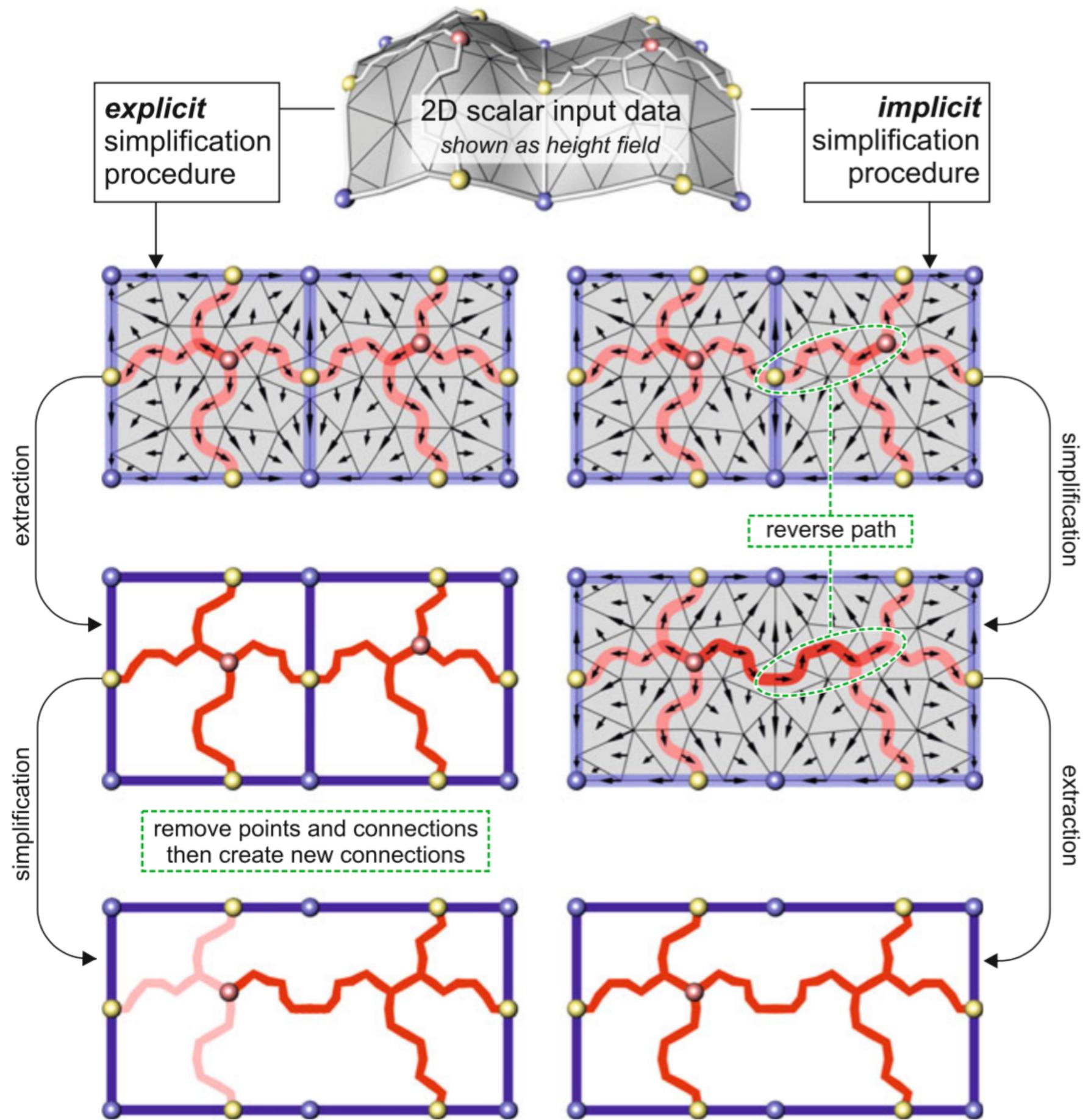
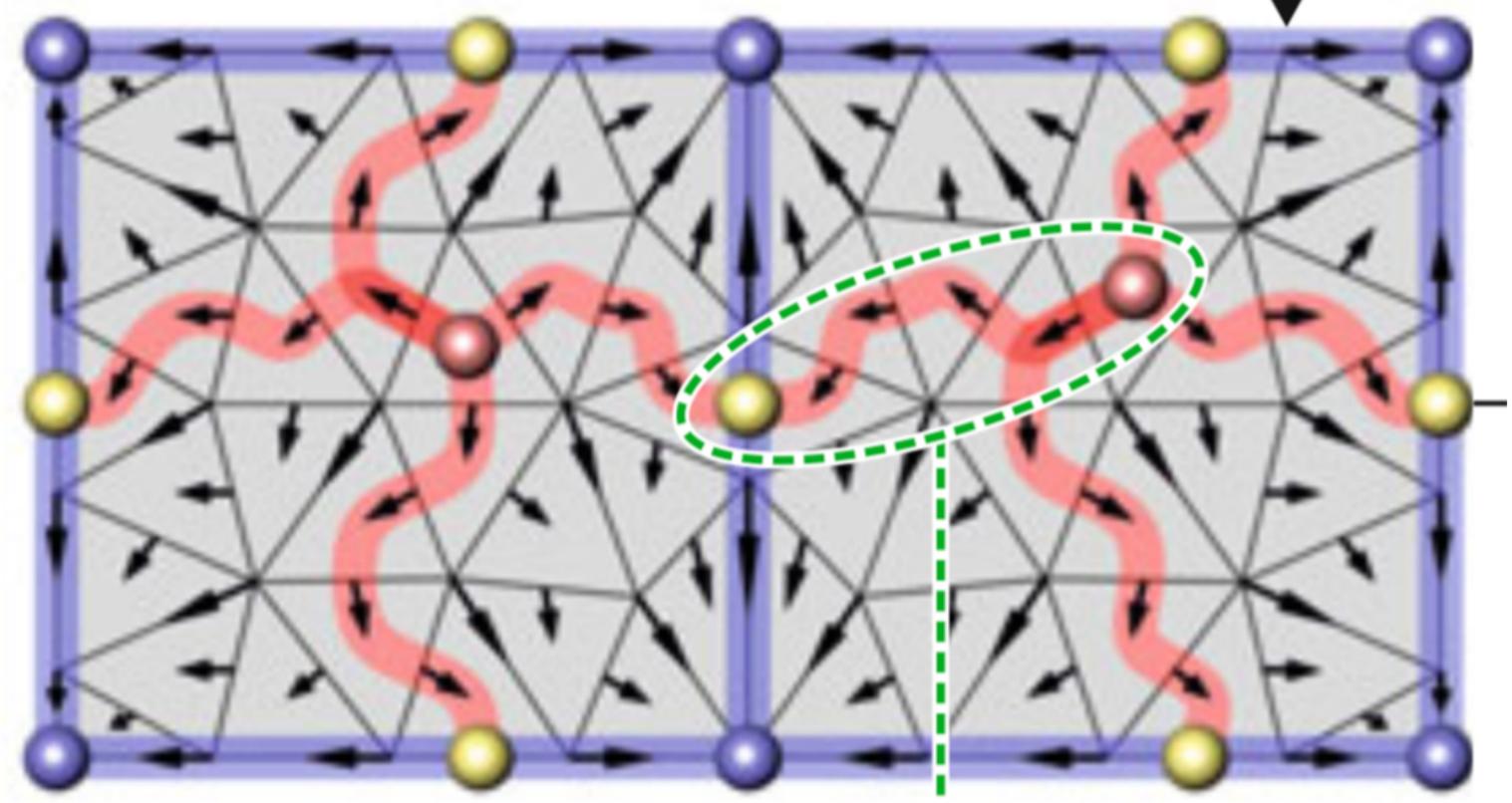


FIGURE 6. Left: A simplicial complex M with a discrete gradient V . The critical cells of V are α , e , f , and v . Right: A refinement N of M and a refinement W of V . The new critical cells $h(\alpha)$ and $h(e)$ are indicated.

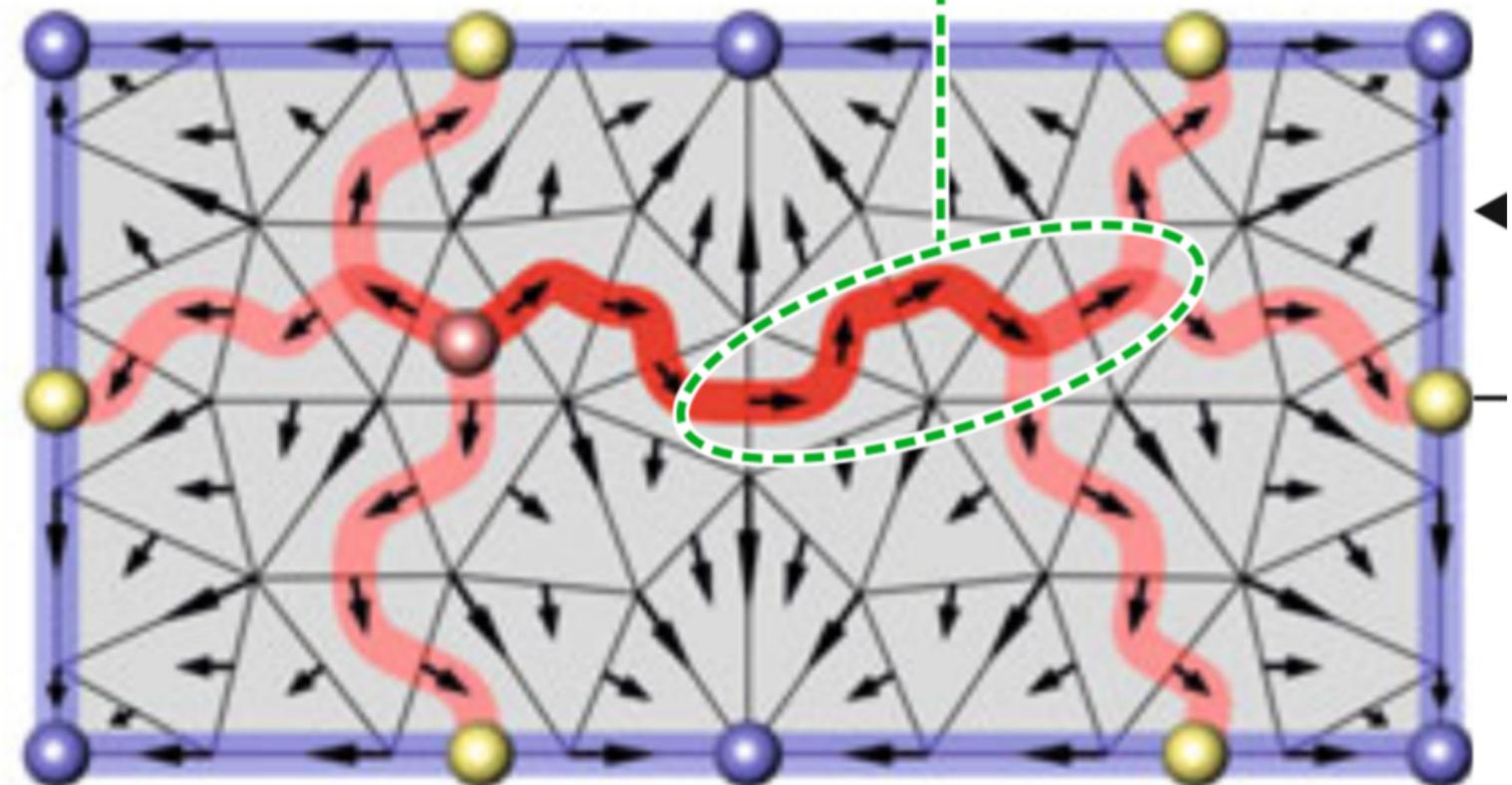
Applications

Simplification of MSC





reverse path



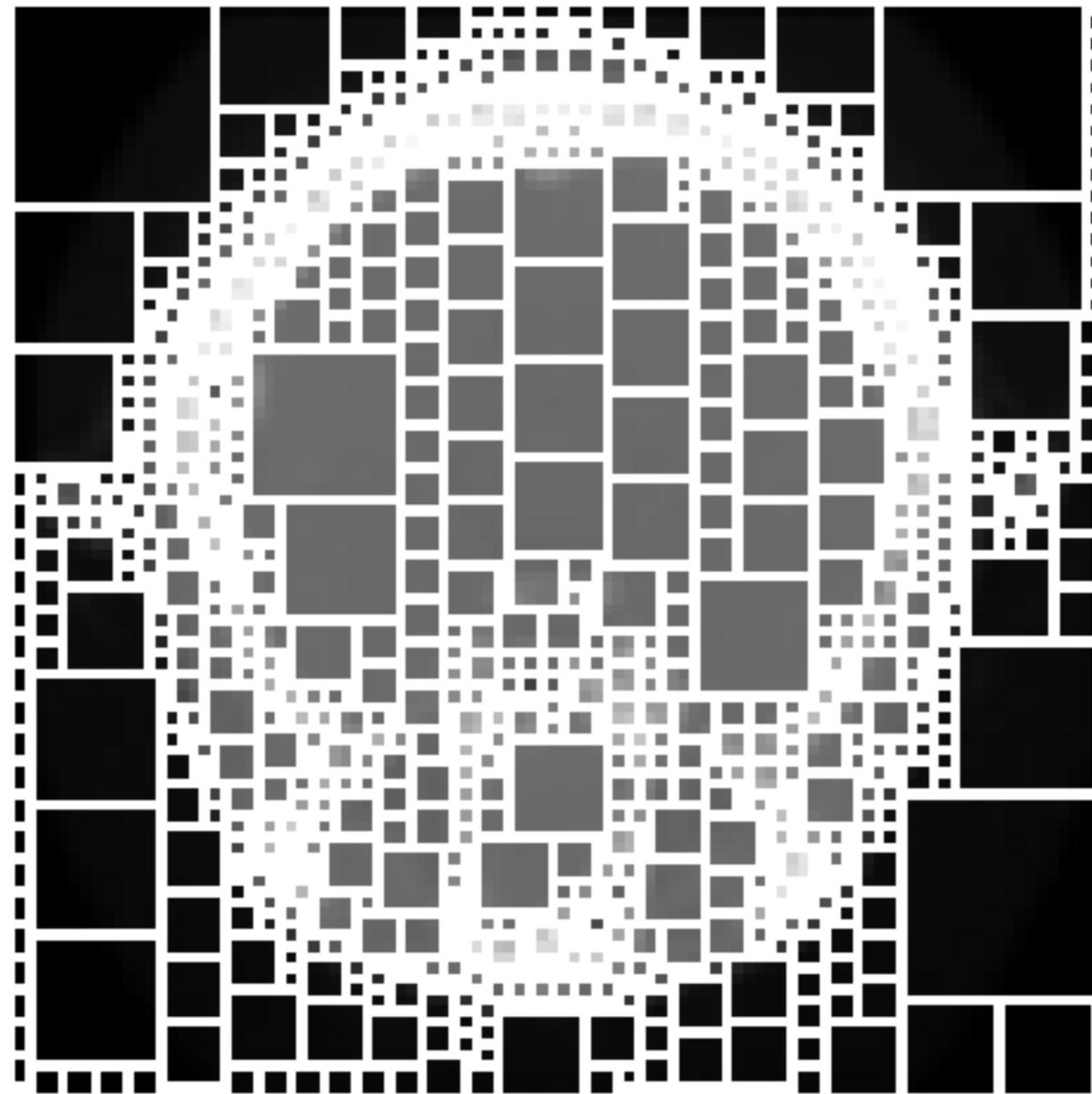


FIGURE 15. Reduced cell decomposition of a head CT scan.



Thanks!

Any questions?

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