Lecture 12: Feb 16, 2017

Lecturer: Prof. Bei Wang <beiwang@sci.utah.edu> *Scribe: Waiming Tai*

This lecture's notes illustrate the concept of computing homology.

12.1 Review of definitions

For any simplicial complex K, we have the following definitions.

Definition 12.1. *The p-th cycle group $Z_p(K)$ is a set of p-th chain $C_p(K)$ with empty boundary. That is, $Z_p(K) = \{c \mid \partial c = 0, c \in Z_p(K)\}$*

Definition 12.2. *The p-th boundary group $B_p(K)$ is a set of p-th chain $C_p(K)$ that is the boundary of a $(p+1)$-th chain. That is , $B_p(K) = \{c \mid c = \partial d \quad \text{for some } d \in C_{p+1}(K)\}$*

Definition 12.3. *The p-th homology group $H_p(K)$ is the p-th cycle group $Z_p(K)$ modulo the p-th boundary group B_p. That is, $H_p(K) = Z_p(K)/B_p(K)$*

From now on, we simplify the notation $Z_p(K) = Z_p$, $B_p(K) = B_p$ and $H_p(K) = H_p$ when K is apparent.

Roughly speaking, H_p is the group of cycles that don't bound. Here is an example.

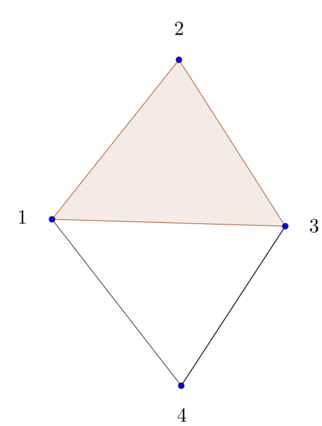

Figure 12.1: The first example

Let $c = 13 + 34 + 14$. c is a cycle which means $c \in Z_1$. However, there is not a $d \in C_2$ such that $c = \partial d$ and so $c \notin B_1$. Therefore, c is a non-identity element of H_1.

Let $c' = 12 + 23 + 13$. $c' \in Z_1$. Also, $c' = \partial d'$ where $d' = 123$ and so $c' \in B_1$. That means c' is an identity in H_1.

Let $c'' = 12 + 23 + 34 + 14$. We can express c'' as $(13 + 34 + 14) + (12 + 23 + 13) = c + c'$. It means that $c'' \approx c$ in H_1.

Here is another example.

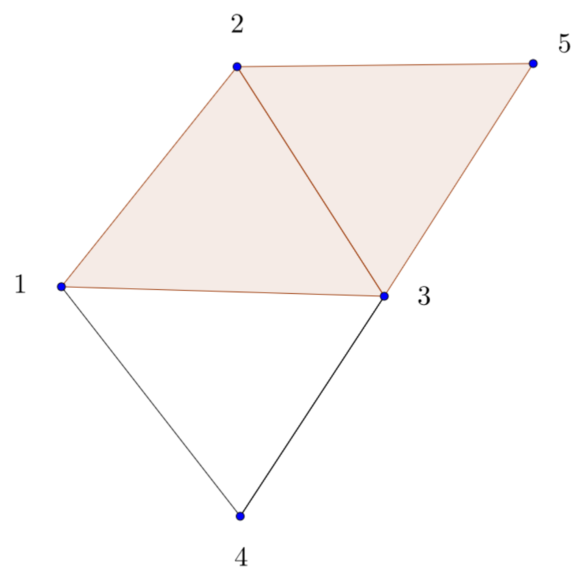

Figure 12.2: The second example

Consider $12 + 25 + 35 + 34 + 14$. Is this cycle an identity in H_1? The answer is yes. We can express it as $(13 + 34 + 14) + (12 + 23 + 13) + (23 + 35 + 25)$. It is easy to see that $12 + 23 + 13$ and $23 + 35 + 25$ are in B_1 but $13 + 34 + 14$ is not.

Definition 12.4. *A generating set of a group G is a subset of G such that every element in G can be expressed as the combination (under group operation) of finitely many elements of the subset and their inverses.*

Definition 12.5. *Rank of a group G $\mathrm{rank}(G)$ is the smallest cardinality of a generating set of G. That is, $\mathrm{rank}(G) = \min_{S \subset G} |S|$ where minimum is over all generating set of G.*

Definition 12.6. *The p-th Betti number β_p is the rank of H_p. That is, $\beta_p = \mathrm{rank}(H_p)$.*

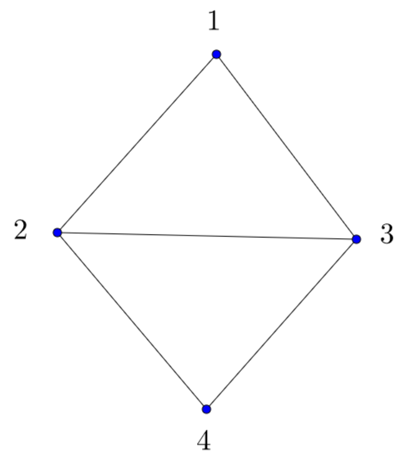

Figure 12.3: Generating set example

In the above example, $rank(H_1) = 2$ not 3. Consider

$$
\begin{aligned}
c_1 &= 12 + 23 + 13 \\
c_2 &= 23 + 34 + 24 \\
c_3 &= 12 + 13 + 34 + 24
\end{aligned}
$$

It is easy to check that the smallest set of H_1 is $\{c_1, c_2\}$ or $\{c_2, c_3\}$ or $\{c_1, c_3\}$. This example also shows that the smallest generating set may not be unique.

Recall that all p-th chain C_p are connected by boundary operator ∂.

$$C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

If $123 \in C_2$, then

$$
\begin{aligned}
\partial(123) &= 12 + 13 + 23 \in C_1 \\
\partial(12) &= 1 + 2 \in C_0
\end{aligned}
$$

More generally,

$$\cdots \to C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \to \dots$$

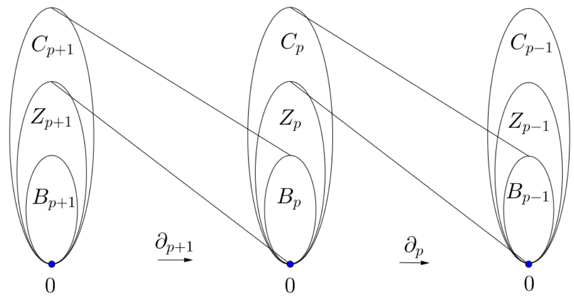

Figure 12.4: Illustration of boundary map

12.2 Reduced homology

Consider the augmentation map $\mathcal{E} : C_0 \to \mathbb{Z}_2$ defined by $\mathcal{E}(u) = 1$ for every vertex u.

$$\cdots \to C_1 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\mathcal{E}} \mathbb{Z}_2 = C_{-1} \xrightarrow{0} 0$$

Definition 12.7. *The p-th reduced homology group $\tilde{H}_p$ is defined as following.*

$$\tilde{H}_p = \ker \partial_p \backslash \operatorname{im} \partial_{p+1} = H_p$$

In particular,

$$\tilde{H}_0 = \ker \mathcal{E} \backslash \operatorname{im} \partial_1$$

Definition 12.8. *The p-th reduced Betti number $\tilde{\beta}_p$ is the rank of $\tilde{H}_p$. That is, $\tilde{\beta}_p = \operatorname{rank}(\tilde{H}_p)$.*

If K is not empty, then

$$\begin{cases} \tilde{\beta}_p = \beta_p & \text{, for } p \geq 1 \\ \tilde{\beta}_0 = \beta_0 - 1 \end{cases}$$

If $K = \emptyset$, then $\tilde{\beta}_{-1} = 1$

12.3 Algorithm

This is the algorithm for computing $\tilde{\beta}_p$.

Input: p-th boundary matrix ∂_p for all p
where the column represent p-simplices, η_p
and the row represent $(p-1)$-simplices, η_{p-1}
Use row and column operation to reduce ∂_p to Smith normal form (SNF) N_p
return $n_0 - n_1$
where n_0 is number of zero column in N_p
and n_1 is number of non-zero row in N_{p+1}

Recall that a matrix is SNF if

- all non-diagonal element are zero
- all non-zero row are above all zero row

Indeed, we can prove that $n_0 = \mathrm{rank}(Z_p)$ and $n_1 = \mathrm{rank}(B_p)$ and therefore the output is exactly $\tilde{\beta}_p$.

Recall that column and row operation consist of the following.

Column operation:

- exchange column k with column l
- add column k to column l

$$\begin{bmatrix} & \vdots & & \vdots & \\ & \vdots & & \vdots & \\ \cdots & \text{col } k & \cdots & \text{col } k + \text{col } l & \cdots \\ & \vdots & & \vdots & \\ & \vdots & & \vdots & \end{bmatrix} = \begin{bmatrix} & \vdots & & \vdots & \\ & \vdots & & \vdots & \\ \cdots & \text{col } k & \cdots & \text{col } l & \cdots \\ & \vdots & & \vdots & \\ & \vdots & & \vdots & \end{bmatrix} \begin{bmatrix} 1 & & & \vdots & \\ \cdots & \ddots & \text{(row } k\text{)} & 1 & \cdots \\ & & \ddots & \text{(col } l\text{)} & \\ & & & \ddots & \\ & & & \vdots & 1 \end{bmatrix}$$

Row operation:

- exchange row k with row l
- add row l to row k

$$\begin{bmatrix} & & \vdots & & \\ \cdots & \cdots & \text{row } k + \text{row } l & \cdots & \cdots \\ & & \vdots & & \\ \cdots & \cdots & \text{row } l & \cdots & \cdots \\ & & \vdots & & \end{bmatrix} = \begin{bmatrix} 1 & \vdots & & & \\ & \ddots & & & \\ & \text{(col } k\text{)} & \ddots & & \\ \cdots & 1 & \text{(row } l\text{)} & \ddots & \cdots \\ & \vdots & & & 1 \end{bmatrix} \begin{bmatrix} & & \vdots & & \\ \cdots & \cdots & \text{row } k & \cdots & \cdots \\ & & \vdots & & \\ \cdots & \cdots & \text{row } l & \cdots & \cdots \\ & & \vdots & & \end{bmatrix}$$

Therefore, $N_p = U_{p-1}\partial_p V_p$ where U_{p-1} represent the row operation and V_p represent the column operation.

Here is the example. The following K is called triangulated 3-ball which consists of all possible combination. That is, $K = \{a, b, c, d, ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd, abcd\}$.

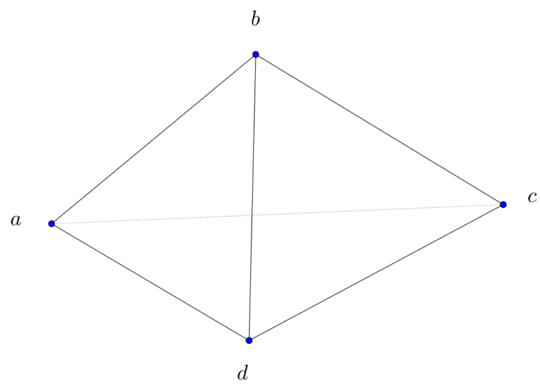

Figure 12.5: Triangulated 3-ball

It is easy to check $\tilde{\beta}_0 = \beta_0 - 1 = 0, \tilde{\beta}_1 = \beta_1 = 0, \tilde{\beta}_2 = \beta_2 = 0$. Now, we can compute this by the above algorithm.

∂_0:

$$\begin{array}{ccccc} & a & b & c & d \\ 1 & 1 & 1 & 1 & 1 \end{array}$$

Adding column 1 to column 2, 3 and 4:

$$\begin{array}{ccccc} & a & b & c & d \\ 1 & 1 & 0 & 0 & 0 \end{array}$$

N_0:

$$\begin{array}{ccccc} & a & b & c & d \\ 1 & 1 & 0 & 0 & 0 \end{array}$$

Therefore, $\mathrm{rank}(Z_0) = 3$ and $\mathrm{rank}(B_{-1}) = 1$.

∂_1:

$$\begin{array}{ccccccc} & ab & ac & ad & bc & bd & cd \\ a & 1 & 1 & 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 1 & 1 & 0 \\ c & 0 & 1 & 0 & 1 & 0 & 1 \\ d & 0 & 0 & 1 & 0 & 1 & 1 \end{array}$$

Adding row 1 to row 2:

$$\begin{array}{ccccccc} & ab & ac & ad & bc & bd & cd \\ a & 1 & 1 & 1 & 0 & 0 & 0 \\ b & 0 & 1 & 1 & 1 & 1 & 0 \\ c & 0 & 1 & 0 & 1 & 0 & 1 \\ d & 0 & 0 & 1 & 0 & 1 & 1 \end{array}$$

Adding column 1 to column 2 and 3:

$$\begin{array}{ccccccc} & ab & ac & ad & bc & bd & cd \\ a & 1 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1 & 1 & 1 & 1 & 0 \\ c & 0 & 1 & 0 & 1 & 0 & 1 \\ d & 0 & 0 & 1 & 0 & 1 & 1 \end{array}$$

Adding row 2 to row 3:

$$\begin{array}{c|cccccc} & ab & ac & ad & bc & bd & cd \\ \hline a & 1 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1 & 1 & 1 & 1 & 0 \\ c & 0 & 0 & 1 & 0 & 1 & 1 \\ d & 0 & 0 & 1 & 0 & 1 & 1 \end{array}$$

Adding column 2 to column 3, 4 and 5:

$$\begin{array}{c|cccccc} & ab & ac & ad & bc & bd & cd \\ \hline a & 1 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 1 & 0 & 1 & 1 \\ d & 0 & 0 & 1 & 0 & 1 & 1 \end{array}$$

Adding row 3 to row 4:

$$\begin{array}{c|cccccc} & ab & ac & ad & bc & bd & cd \\ \hline a & 1 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 1 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

N_1:

$$\begin{array}{c|cccccc} & ab & ac & ad & bc & bd & cd \\ \hline a & 1 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 1 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Therefore, $\mathrm{rank}(Z_1) = 3$ and $\mathrm{rank}(B_0) = 3$. Also, $\tilde{\beta}_0 = 0$.

∂_2:

$$\begin{array}{c|cccc} & abc & abd & acd & bcd \\ \hline ab & 1 & 1 & 0 & 0 \\ ac & 1 & 0 & 1 & 0 \\ ad & 0 & 1 & 1 & 0 \\ bc & 1 & 0 & 0 & 1 \\ bd & 0 & 1 & 0 & 1 \\ cd & 0 & 0 & 1 & 1 \end{array}$$

Adding row 1 to row 2 and 4:

$$\begin{array}{c|cccc} & abc & abd & acd & bcd \\ \hline ab & 1 & 1 & 0 & 0 \\ ac & 0 & 1 & 1 & 0 \\ ad & 0 & 1 & 1 & 0 \\ bc & 0 & 1 & 0 & 1 \\ bd & 0 & 1 & 0 & 1 \\ cd & 0 & 0 & 1 & 1 \end{array}$$

Adding column 1 to column 2:

$$\begin{array}{c|cccc} & abc & abd & acd & bcd \\ \hline ab & 1 & 0 & 0 & 0 \\ ac & 0 & 1 & 1 & 0 \\ ad & 0 & 1 & 1 & 0 \\ bc & 0 & 1 & 0 & 1 \\ bd & 0 & 1 & 0 & 1 \\ cd & 0 & 0 & 1 & 1 \end{array}$$

Adding row 2 to row 3, 4, 5 and 6:

$$\begin{array}{lllll} & abc & abd & acd & bcd \\ ab & 1 & 0 & 0 & 0 \\ ac & 0 & 1 & 1 & 0 \\ ad & 0 & 0 & 0 & 0 \\ bc & 0 & 0 & 1 & 1 \\ bd & 0 & 0 & 1 & 1 \\ cd & 0 & 0 & 1 & 1 \end{array}$$

Adding column 2 to column 3:

$$\begin{array}{lllll} & abc & abd & acd & bcd \\ ab & 1 & 0 & 0 & 0 \\ ac & 0 & 1 & 0 & 0 \\ ad & 0 & 0 & 0 & 0 \\ bc & 0 & 0 & 1 & 1 \\ bd & 0 & 0 & 1 & 1 \\ cd & 0 & 0 & 1 & 1 \end{array}$$

Exchanging row 3 with row 4:

$$\begin{array}{lllll} & abc & abd & acd & bcd \\ ab & 1 & 0 & 0 & 0 \\ ac & 0 & 1 & 0 & 0 \\ bc & 0 & 0 & 1 & 1 \\ ad & 0 & 0 & 0 & 0 \\ bd & 0 & 0 & 1 & 1 \\ cd & 0 & 0 & 1 & 1 \end{array}$$

Adding row 3 to row 5 and 6:

$$\begin{array}{lllll} & abc & abd & acd & bcd \\ ab & 1 & 0 & 0 & 0 \\ ac & 0 & 1 & 0 & 0 \\ bc & 0 & 0 & 1 & 1 \\ ad & 0 & 0 & 0 & 0 \\ bd & 0 & 0 & 0 & 0 \\ cd & 0 & 0 & 0 & 0 \end{array}$$

Adding column 3 to column 4:

$$\begin{array}{lllll} & abc & abd & acd & bcd \\ ab & 1 & 0 & 0 & 0 \\ ac & 0 & 1 & 0 & 0 \\ bc & 0 & 0 & 1 & 0 \\ ad & 0 & 0 & 0 & 0 \\ bd & 0 & 0 & 0 & 0 \\ cd & 0 & 0 & 0 & 0 \end{array}$$

N_2:

$$\begin{array}{lllll} & abc & abd & acd & bcd \\ ab & 1 & 0 & 0 & 0 \\ ac & 0 & 1 & 0 & 0 \\ bc & 0 & 0 & 1 & 0 \\ ad & 0 & 0 & 0 & 0 \\ bd & 0 & 0 & 0 & 0 \\ cd & 0 & 0 & 0 & 0 \end{array}$$

Therefore, $\mathrm{rank}(Z_2) = 1$ and $\mathrm{rank}(B_1) = 3$. Also, $\tilde{\beta}_1 = 0$.

∂_3:

$$\begin{array}{cc} & abcd \\ abc & 1 \\ abd & 1 \\ acd & 1 \\ bcd & 1 \end{array}$$

Adding row 1 to row 2, 3 and 4:

$$\begin{array}{cc} & abcd \\ abc & 1 \\ abd & 0 \\ acd & 0 \\ bcd & 0 \end{array}$$

N_3:

$$\begin{array}{cc} & abcd \\ abc & 1 \\ abd & 0 \\ acd & 0 \\ bcd & 0 \end{array}$$

Therefore, $\mathrm{rank}(Z_3) = 0$ and $\mathrm{rank}(B_2) = 1$. Also, $\tilde{\beta}_2 = 0$.

Here is another example. This example is same as the previous one except that the center is hollow. That is, $K = \{a, b, c, d, ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd\}$.

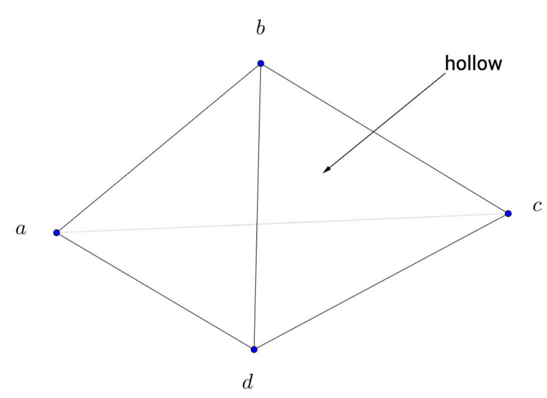

Figure 12.6: Hollow triangulated 3-ball

In this case, ∂_3 doesn't exist. Therefore, $\tilde{\beta}_2 = 1 - 0 = 1$.