CS 6170 Computational Topology: Topological Data Analysis **University of Utah**
Spring 2017 **School of Computing**

Lecture 7: Jan 31, 2017

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This lecture's notes illustrates the concept of Manifolds and describes Homology.

7.1 Some definitions related to Simplical Complex

Whether it is Rips Complex or Cech Complex, as the radius of ball increases, the complex gets bigger. All smaller complex formed along the way are include in the bigger complexes.

This observation gives rise to the concept of filtration.

Definition 7.1. *A filtration is a sequence of Simplical Complexes connected by inclusion.* $\phi \subseteq K_0 \subseteq K_1 \subseteq K_2 \cdots \subseteq K_n$

Here, left simplical complex is subset of right. This idea is important for Persistent Homology.

7.2 Topological Equivalence and Homeomorphism

Previous lecture notes cover Homotopy Equivalence. Similarly, Here we are going to describe Topological Equivalence.

Topological spaces are Homeomorphic or Topologically equivalent if there exists a continuous bijection from one space to the other and whose inverse is also continuous.

A very trivial example is of Donout and Coffee Cup as shown is Figure 7.1:

Figure 7.1: Continuous deformation of coffee cup to donut [1]

Definition 7.2. *A function $f : \mathbb{X} \to \mathbb{Y}$ between two topological spaces is a homeomorphic if it has following properties:*

1. *f is a bijection (It is both one-to-one and onto)*

2. *f is continuous*

3. *The inverse f^{-1} is continuous*

7.2.1 Review (one-to-one and onto)

1. **one-to-one (Injective)**

   This is a function that preserves distinctness: it never maps distinct elements of its domain to the same element of its target set. Consider sets X and Y such. Here X is known as domain, and Y is known as target set (Range). Let there be a $f : X \to Y$ such that every element of X is mapped to a unique element in Y. This is depicted in Figure 7.2

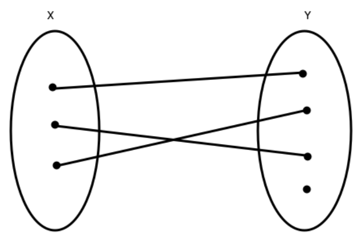


Figure 7.2: Injective Mapping

2. **onto (Surjective)**

   This is a function that maps every element in domain to an element in range. Thus, every element in range has a corresponding element in domain Let there be a $f : X \to Y$ such that every element of X is mapped to some element in Y. This is depicted in Figure 7.3

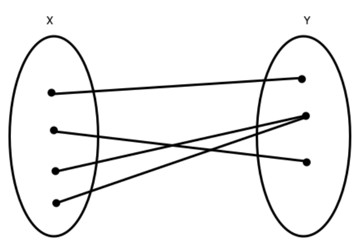


Figure 7.3: Surjective Mapping

3. **Bijective (both Injective and Surjective)**

   This is a function that has properties of both Injective and Surjective functions. This is depicted in Figure 7.4

7.2.2 Homology

1. **0-Dimension:** Counts number of connected components

2. **1-Dimension:** Counts number of tunnels.

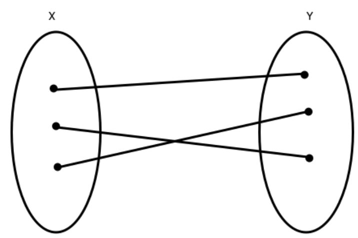


Figure 7.4: Bijective Mapping

NOTE: If two topological spaces have same homology, then they may or may not be homeomorphic. But if two topological spaces have different homology then they are not homeomorphic.

NOTE: Computational Topology cares for underlying structures and properties that do not change on continuous deformation. Where as Computational geometry looks for curvature, angles and similar things.

7.2.3 Example

Consider a unit interval (0,1) $\mathbb{X}$ and a circle $\mathbb{Y}$ as shown in Figure 7.5. Task is to determine if both of them are homeomorphic or not.

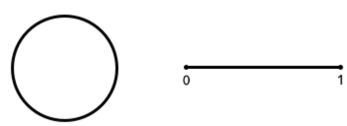


Figure 7.5: Circle and interval (0,1) are not homeomorphic

1. In $\mathbb{X}$ both points 0 and 1 map to the same point in $\mathbb{Y}$. So this mapping is no longer bijective. This violates first property for two topological spaces to be homeomorphic. So, both of theses spaces are not homeomorphic.
2. We can also look at homology of two spaces:

Homology	Unit Interval	Circle
Dim-0 (Connected Components)	1	1
Dim-1 (Tunnels)	0	1

Table 7.1: Homology of Unit Interval and Circle

Since both of these have different homology, they are not Homeomorphic / Topologically equivalent.

Exercise: Can u define a function $f : \mathbb{X} \to \mathbb{Y}$ that is continuous and bijective? (in short satisfies all three properties for being topological equivalent)

7.3 Manifold (ant's world)

1. **1-Manifold** Assume an ant over a line. The local neighbourhood of ant is 1-dimensional. This represents 1-Manifold. It is represented in Figure 7.6

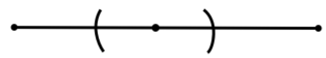


Figure 7.6: 1D-Manifold

2. **2-Manifold** Assume an ant over a piece of donut. The local neighbourhood of ant is 2-dimensional (plane). This represents 2-Manifold. It is represented in Figure 7.7

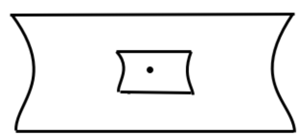


Figure 7.7: 2D-Manifold

3. **3-Manifold** Assume an ant inside solid snow ball. The local neighbourhood of ant is 3-dimensional. This represents 3-Manifold. It is represented in Figure 7.8. However, the surface of snow ball is 2-Manifold.

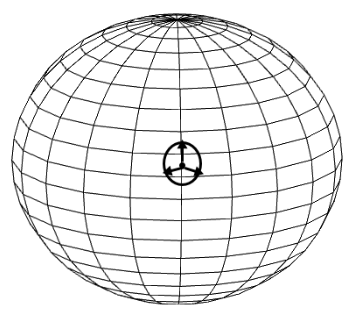


Figure 7.8: 3D-Manifold

7.3.1 2-Manifold without Boundary

Open Unit Disk: $D = \{\chi \in \mathbb{R}^2 | \, \|\chi\| < 1\}$

Claim: D is homeomorphic to $\mathbb{R}^2$ (Entire Plane)

Proof: Define a map / function $f : D \to \mathbb{R}^2$

$f(\chi) = \frac{\chi}{1-\|\chi\|}$

If χ is on boundary of disk f maps to infinity.

NOTE: Any open disk is homeomorphic to D as shown in figure 7.9.

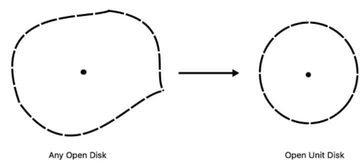


Figure 7.9: Any open disk is homeomorphic to open unit disk

Definition 7.3. *A 2- Manifold without boundary is a topological space $\mathbb{M}$ whose points all lie in open disks. Intuitively, this means that $\mathbb{M}$ locally looks like a plane.*

This is shown in Figure 7.10 where each point on surface of donut (2- Manifold) lies in open disk.

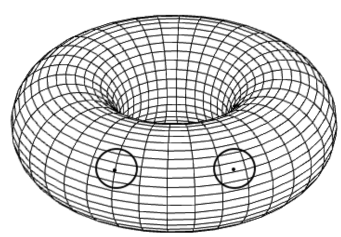


Figure 7.10: All points on torus lie in open disks

Definition 7.4. *We get 2-Manifold with boundary by removing open disks from 2-Manifold without boundary.*

This is depicted in figure 7.11 and 7.12.

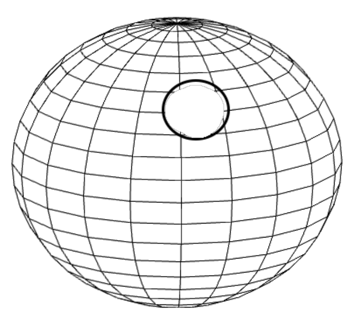


Figure 7.11: Removing open disks from sphere to obtain 2-Manifold with boundary

Examples of manifold without boundary:

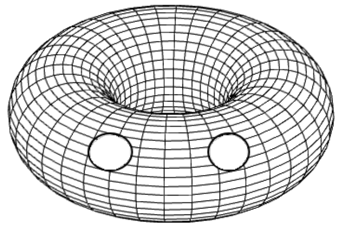


Figure 7.12: Removing open disks from torus to obtain 2-Manifold with boundary

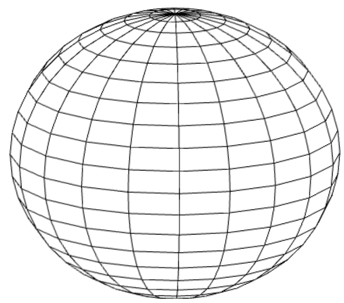


Figure 7.13: Sphere

1. Sphere (Figure 7.13)
2. Torus (Figure 7.14)

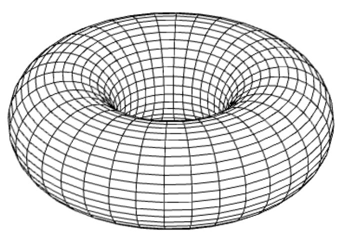


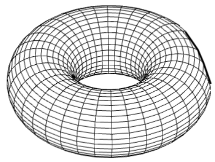
Figure 7.14: Torus

3. Double Torus (Figure 7.15 and 7.16)

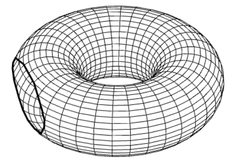
7.3.2 2-Manifold with Boundary

Examples of manifold with boundary:

1. Disk (Figure 7.17)



(a) Left Part



(b) Right Part

Figure 7.15: Cutting out open disks from 2 Torus' and gluing them together to form double torus

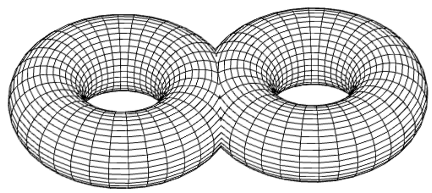


Figure 7.16: Double Torus

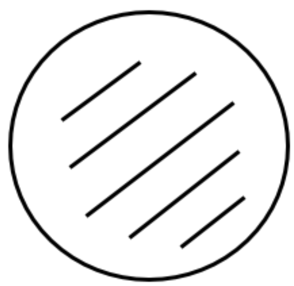


Figure 7.17: Disk

It has one boundary.

2. Cylinder (Figure 7.18) It has 2 boundaries (one at top and one at bottom).

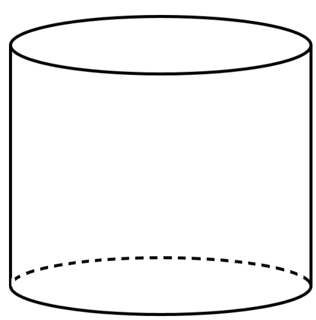


Figure 7.18: Cylinder

3. Mobius Strip (Figure 7.19)

It has 1 boundary (1 connected Component at top).

Globally mobius strip has 1 side, locally it has 2 sides. Sphere also has 2 sides.

If you cut along the mobius strip 1st time you get a circle. If you cut second time at the intersection you get 2 circles linked like a chain. They can't be separated without cutting one of them.

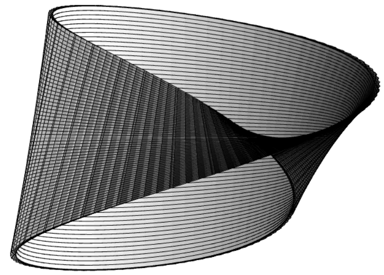


Figure 7.19: Mobius Strip

7.4 Orientablility

- **Mobius strip** It is an example of non-orientable surface: 2 sides locally and 1 side globally. Taking an oriented loop around mobius strip changes its orientation as shown in Figure 7.20.

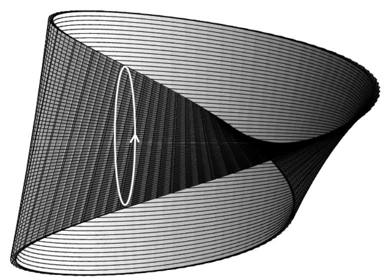


Figure 7.20: Orientation of mobius strip

- **Torus** Orientable surface Consider a small loop clutching the handle. Moving the loop around does not change the orientation of the loop as shown in figure 7.21.

Definition 7.5. *If all closed curves in a two manifold are orientation preserving, then the 2-manifold is orientable.*

7.4.1 Creating compact 2-Manifold by Polygonal Schema

1. **Sphere from Polygon Schema** as shown in Figure 7.22 (a)
2. **Torus from Polygon Schema** Glue a, a together, b, b together preserving orientation as shown in Figure 7.22(b)

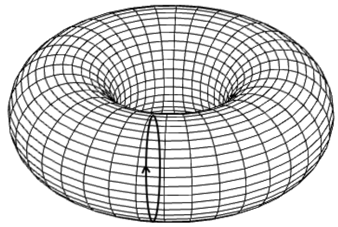
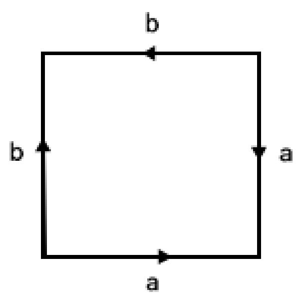


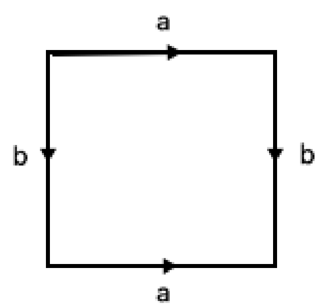
Figure 7.21: Orientation of Torus

3. **Projective Plane from Polygon** IP^2 Glue a disc to a mobius strip. It can't be embedded in IR^3 without self-intersection. Polygone schema for Projective Plane is shown in Figure 7.22(c)

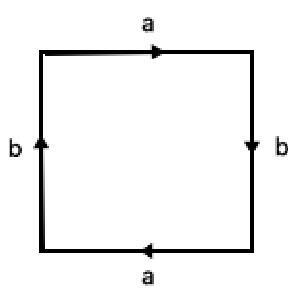
4. **Klein Bottle from Polygon Schema** Glue 2 mobius strips together. Its polygon schema is depicted in Figure 7.22(d) Actual Klein Bottle is shown in Figure 7.23. Imagine joining the two ends of this tube after joining (a,a) but in such a way that the orientation of the boundaries is preserved.



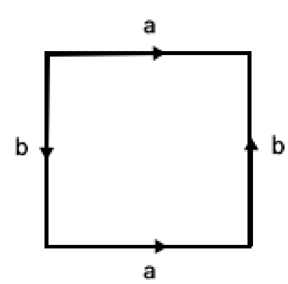
(a) Polygon Schema Square



(b) Polygon Schema Torus



(c) Polygon Schema Projective Plane



(d) Polygon Schema Klein Bottle

Figure 7.22: Polygon Schemas

Theorem 7.6. *Classification theorem for compact 2-manifold. The two infinite families:*

- $\mathbb{S}^2$, $\mathbb{T}^2$, $\mathbb{T}^2 \# \mathbb{T}^2$, ...
  and

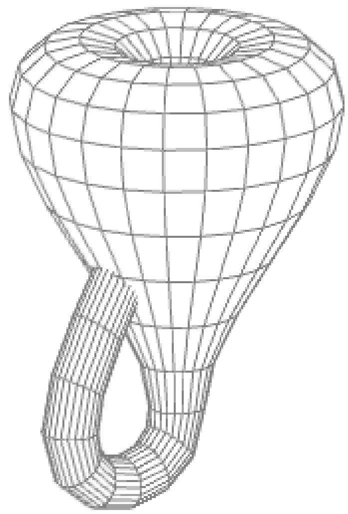
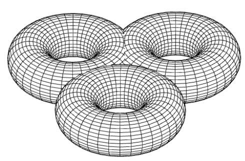


Figure 7.23: Klein Bottle

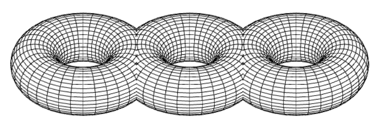
- $\mathbb{P}^2$, $\mathbb{P}^2$ # $\mathbb{P}^2$, ...

exhaust the compact 2-manifolds without boundary.

One Family of Torus' (one member) is shown in Figure 7.24.



(a) 3 Torus combined together non collinearly



(b) 3 Torus combined together collinearly

Figure 7.24: Two examples of $\mathbb{T}^2$ # $\mathbb{T}^2$ # $\mathbb{T}^2$

Definition 7.7. $\mathbb{M}$ *is compact if for every cover of* $\mathbb{M}$ *by open set* $\mathbb{S}$*, open cover, we can find a finite number of sets that cover* $\mathbb{M}$*.*

Claim: A subset of Euclidean space is compact if it is closed and bounded [contained in a ball of finite radius]

- Think of patches covering a surface. If we can find a finite number of patches that cover the surface, then it is bounded.

7.4.2 Homology

Historically Homology asks, can two spaces be distinguished by examining their holes?.

Betti Numbers:

1. β_0 # number of connected components
2. β_1 # of tunnels
3. β_2 # of voids

Topological Space	β_0	β_1	β_2
Disk	1	1	0
Sphere	1	0	1
Torus	1	2	1
Double Torus	1	4	1

Table 7.2: Betti Numbers

NOTE: We are talking about the number of independent CC—tunnels—voids **i.e.** set of CC—tunnels—voids that generate all other possible CC—tunnels—voids.

References

[1] HENRY SEGERMAN "Topology Joke," Shapeways