CS 6170 Computational Topology: Topological Data Analysis **University of Utah**
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6.1 Complexes

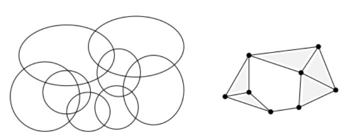
Simplicial complexes convey common patterns emerging from the intersection of sets within topological spaces In the case of geometric spaces simplicial complexes may be used to represent the geometric structure of the underlying space. In order to gain a general understanding of the entire simplicial complex it is useful to study a subset to the simplicial complex which is common to all sets which compose the complex, denoted the *nerve* of the simplicial complex..

Definition 6.1. *Given $\mathcal{F}$, a finite collection of sets, the* **nerve** *of $\mathcal{F}$ is a subset of sets X subset to $\mathcal{F}$ whose intersection is non-empty. More formally:*

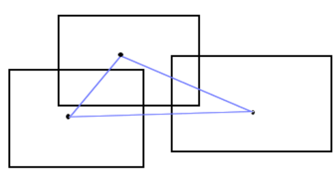
$$\textit{Nerve}(\mathcal{F}) = \{X \subseteq \mathcal{F} | \cap X \neq \emptyset\}$$

First we considerd $\mathcal{F}$ to consist of convex sets. Recall a convex set is a set of points such that any line connecting two points contained in the set will also be contained within the set. It is also usefull to note that the intersection of convex sets remains convex.

Not needing to assume subsets of $\mathcal{F}$ are convex we know the nerve of $\mathcal{F}$ will remain an abstract simplicial complex. In the case we are considering some geometric space with some covering C then the nerve of C often possess the underlying geometry of the space. For example if we consider a covering as shown in figure 6.1 each set can be associated to a vertex and intersections given by adjoining lines.



(a) 2-dimensional nerve of 8 set covers



(b) 2-dimensional nerve of covering with 3 rectangular sets

Figure 6.1: Nerve of 2-dimensional set covers

Background needed before procceeding is a weaker notion of topological equivalence which still affords a notion of equivalence between topological spaces, denoted **homotopy type**, where **homotopy** is given by definition 6.2.

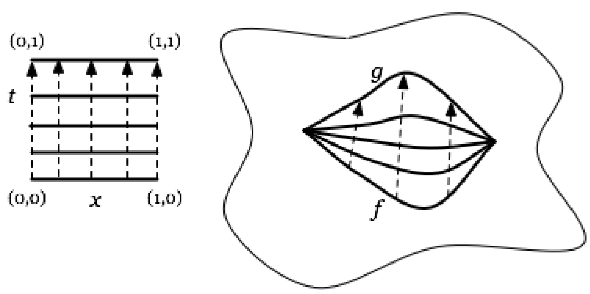
Definition 6.2. *The continuous map $H : \mathbb{X} \times [0,1] \to \mathbb{Y}$ is the* ***homotopy*** *for the two continuous maps $f, g : \mathbb{X} \to \mathbb{Y}$ if for $t = 0$ then $H(x, 0) = f(x)$ and for $t = 1,\ H(x, 1) = g(x)\ \forall x \in \mathbb{X}$.*

We can extend the notion of homotopy to allow a means of correlating two nested topological spaces. Two topological spaces have *homotopy equivalence* if there exists two continuous functions between both spaces in both directions whose composition affords an identity an *identity map*.

Definition 6.3. *Given two continuous maps $f : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{Y} \to \mathbb{X}$ over the topological spaces $\mathbb{X}$ and $\mathbb{Y}$ then map f is called a* ***homotopy equivalence*** *if the mutual compositions of f and g are homotopic. More formally, f is a homotopy equivalence if $f \circ g \simeq \mathbb{I}_{\mathbb{Y}}$ and $g \circ f \simeq \mathbb{I}_{\mathbb{X}}$.*

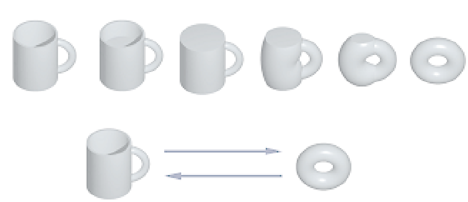
Conceptually we may view the homoptopy between two topological spaces as a time series of functions, $f_t : \mathbb{X} \to \mathbb{Y}$ and defined by $f_t(x) = H(x, t)$, whose initial state is $f_0 = g$ and continuously transforms to $f_1 = g$. Visually we can see two functions γ_1 and γ_2 are homotopic, $\gamma_1 \simeq \gamma_2$, in $\mathbb{R}^2$ by the following series of transformations shown in figure 6.2 which enforces the conditions that for $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^2$ we have $\gamma_1[0] = \gamma_2[0]$ and $\gamma_1[1] = \gamma_2[1]$

Figure 6.2: Deformation of two homotopic functions in $\mathbb{R}^2$



Homotopic equivalence then allows an equivilence relation between toplogical spaces $\mathbb{X}$ and $\mathbb{Y}$, denoted $\mathbb{X} \simeq \mathbb{Y}$, which catagorises homotopically equivalent spaces as belonging to the same **homotopy type**. To answer the well known joke as to why topologists can't tell the difference between their coffee and doughnuts, it is because both spaces, the mug and torus, have homotopy equivalence since one can be deformed into the other as illustrated in figure 6.3.

Figure 6.3: Why coffee mugs are doughnuts



When again considering a finite collection of subsets within $\mathcal{F}$, the nerve of $\mathcal{F}$ maintains an important attribute in that it preserves homotopic equivalence.

Theorem 6.4. ***Nerve Theorem:*** *Let $\mathcal{F}$ be a finite collection of closed convex sets in Euclidean space. The nerve of $\mathcal{F}$ and the union of sets in $\mathcal{F}$ neccesarily have the same* ***homotopy type***.

The nerve theorem tells us that if $\cup\mathcal{F}$ is triangulable, with all subsets closed and all non-empty common intersections contractible, then we have $Nerve(\mathcal{F}) \simeq \cup\mathcal{F}$. Therefore if all sets belong to $\mathbb{R}^d$ then the subcollection of $k \geq d+1$ sets will not have all $\binom{k}{d+1}$ combonations of d-simplicies in the nerve without having the entire k-simplex.

6.1.1 Ĉech Complex

An important special case when considering simplicial complexes beyond all convex sets are specifically sets which are closed geometric balls. We denote such balls as $B_x(r)$ signifying the set of points in $\mathbb{R}^d$ centered at x and contained by the closed cover with radius r.

Definition 6.5. *The* ***Ĉech complex*** *of S with radius r is the nerve of the union of balls parameterised by r.*

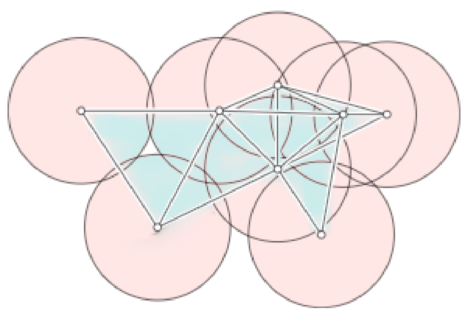
$$\hat{C}ech(r) = \{\sigma \subseteq S \mid \bigcap_{x\in\sigma} B_x(r) \neq \emptyset\}$$

Claim 6.6. *Given $r_0 \leq r_1$ then $\hat{C}ech(r_0) \sqsubseteq \hat{C}ech(r_1)$.*

It is intuitive to see that claim 6.6 is a result of the fact that as you increase r_0 there are more intersections between $B_x(r_0)$ and $B_x(r_1)$ resulting in the inclusion of the Ĉech simplex Ĉech(r_0) in Ĉech(r_1).

To demonstrate the Ĉech complex and it's accompanying nerve consider the nine points in figure 6.4. The Ĉech complex then finds the pairwise intersections between disk centers which are connected by straight edges to form the nerve of the collection of balls. The Ĉech complex then fills nine of the ten possible triangles as well as the two tetrahedra.

Figure 6.4: Ĉech Complex of nine points in $\mathbb{R}^2$.



Instead of checking all subcollections for non-empty common intersections, we may just check pairs and add 2 and higher-dimensional simplices whenever we can. The resulting approach leads us to the Vietoris-Rips complex. The only difference between the Vietoris-Rips and the Ĉech complexes in figure 6.4 would be the tenth triangle which, as we will see in the next section, would only be included in the Vietoris-Rips complex.

6.1.2 Vietoris-Rips Complexes

As opposed to considering all subcollections we may also focus only on intersections of sets, adding 2 and higher dimensional simplicies whenever edges are contained within the complex. The result is the Vietoris-Rips complex of set S and r being all subsets of diameter at most $2r$.

Definition 6.7. *The* ***Vietoris-Rips Complex*** *of* S*, parameterised by* r*,consists of 2 and higher dimensional pairwise connections of subsets of diameter at most* $2r$*.*

$$\textit{Vietoris-Rips(r)} = \{\sigma \subseteq S \mid diam(\sigma) \leq 2r\}$$

Since the Vietoris-Rips complex amounts to edges equivalent to those of the Ĉech complex it is easy to see the Ĉech complex contains every simplex of the Vietoris-Rips complex. For this reason Ĉech(r) $\subseteq$ Vietoris-Rips(r). The containment of the Ĉech complex within the Vietoris-Rips complex can be reversed by increasing the radius defined in the Ĉech complex by some multiplative constant.

Lemma 6.8. *Given* S*, a finite set of points in* $\mathbb{R}^2$ *and* $r \geq 0$*, then Ĉech(r)* $\sqsubseteq$ *Vietoris-Rips(r)* $\sqsubseteq$ *Ĉech(*$\sqrt{2}r$*)*

Lemma 6.8 leads us to Theorem 6.9 which shows we are able to paramaterise the correlation between Ĉech and Vietoris-Rips complexes through the dimensionality of the topological space under question and ball radius.

Theorem 6.9. *Let* X *be a set of points in* $\mathbb{R}^d$*,* $C_\epsilon(X)$ *be the Ĉech complex of the cover of* X *by balls of radius* $\epsilon/2$*, and* $VR_{epsilon}(X)$ *the Vietoris-Rips cover of* X*, then there is a chain of inclusions satisfying*

$$VR_{\epsilon'}(X) \sqsubseteq C_\epsilon(X) \sqsubseteq VR_\epsilon(X) \textit{ where } \frac{\epsilon}{\epsilon'} \geq \sqrt{\frac{2d}{d+1}}$$

Recent works in which the use of Ĉech and Vietoris-Rips complexes have proven beneficial have been in the problem of network coverage. Silvia and Ghrist show this in their analysis of maximizing coverage through signal strength, equating to ball radius, and number of network nodes in order to minimize "dead" spaces as demonstrated in figure 6.5 [SilvaGhrist].

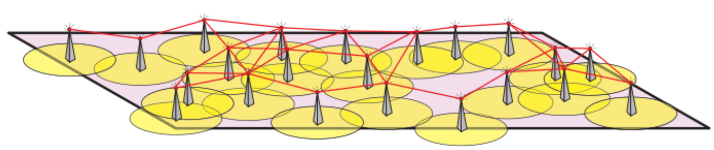


Figure 6.5: Use of Ĉech complex nerve (given in red) to study network coverage [SilvaGhrist].

Biographical Notes: The concept of nerve was introduced by Alexandrov [Alexandrov]. The nerve theorem by Borsuk [Borsuk] and Leray [Leray]. Ĉech complexes originated from the theory of Ĉech homology. Vietoris-Rips complex was introdices by Vietoris [Vietoris] as well as Rips [Rips].

6.1.3 Delaunay Complexes

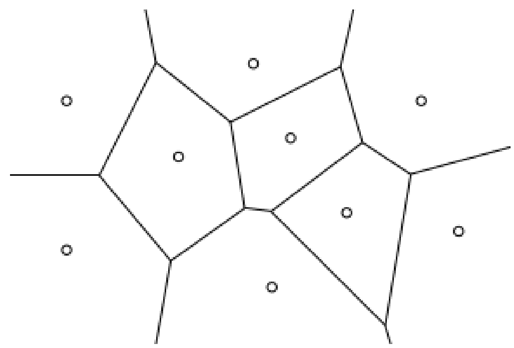
The Delaunay complex deviates from complexes previously mentioned in that as apposed to convex ball covers we may also use *Voronoi cells* as our convex cover.

Definition 6.10. *A* ***Voronoi Cell*** *of a point u in $S \subseteq \mathbb{R}^d$ is the set of points for which u is the closest.*

$$V(X) = \{x \in \mathbb{R}^d \mid ||x - u|| \leq ||x - v|| \; \forall v \in S\}$$

If we are to consider $\mathbb{R}^2$, with only one point in S then $V(u) = \mathbb{R}^2$, i.e. the entire space. If we consider two points $u, v = \mathbb{R}^2$ the Voronoi cells are the half plane separated by the perpendicular bisector of the line segment connecting u and v. For three points first consider the vertex circumcenter to the three points. The Voronoi cell is then the space contained within the bisecting lines from the circumcenter to each point.

Figure 6.6: Voronoi cells formed for 9 points



Generalized to higher dimensions the Voronoi cell V_u of point u is a polyhedron in $\mathbb{R}^d$ known to be convex as a result of being the intersection of convex half spaces.

Definition 6.11. *$V(u)$ is the intersection of half spaces of points at least as close to u as to v for any v in S.*

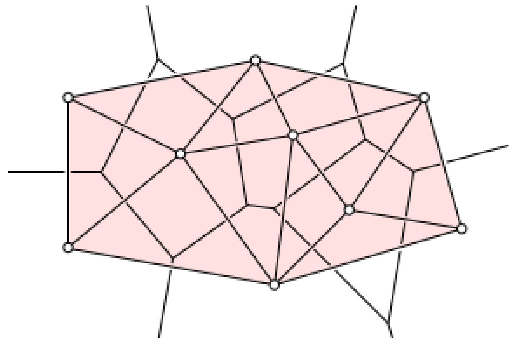
The nerve arising from the use of Voronoi cells allows for a geometric constuction which limits the dimension of the resulting simplicies.

Definition 6.12. *The* ***Delaunay Complex****, or* ***Delaunay Triangulation****, of a finite set $S \subseteq \mathbb{R}^d$ is isomorphic to the nerve of the Voronoi diagram*

$$\textit{Delaunay} = \{\sigma \subseteq S \mid \bigcap_{u \in \sigma} V_u \neq \emptyset\}$$

As shown in figure 6.7 we obtain the Delaunay triangulation from the convex hull formed from the nerve of the Vornoi diagram which connects points by line segments to neighboring points whose Voronoi cells are adjacent.

Figure 6.7: Delaunay covering imposed over Voronoi cells in $\mathbb{R}^2$. Image from [EdelHarer].



Biographical Notes: Voronoi diagrams originated by Georgy Voronoi [Voronoi1][Voronoi2]. Delaunay triangulations are the result of Boris Delaunay [Delaunay]

6.1.4 Alpha Complexes

Combining the previously introduced notion of a ball covering over S, where $B_u(r)$ is the ball of radius r centered at $u \in S$ with the Voronoi covering we are able to define $R_u(r)$ as all points $u \in S$ where the ball covering intersects the Voronoi cell containing u, i.e. $R_u(r) = V(u) \cap B_u(r)$.

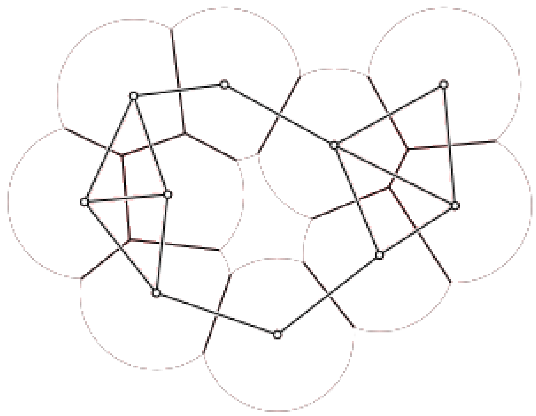
Definition 6.13. *Given the Voronoi cell $V(u)$ covering $u \in S$ and ball of radius r $B_u(r)$ and defining $R_u(r) = V(u) \cap B_u(r)$ the α-**Complex** of radius r is the subcomplex of the Delaynay Complex, denoted*

$$Alpha(r) = \{\sigma \subseteq S \mid \bigcap_{u\in\sigma} R_u(r) \neq \emptyset\}$$

Since the α-complex uses the union of convex closed balls centered at each point of some finite set $S \subseteq \mathbb{R}^d$ intersected with the corresponding, also convex, Voronoi cells of S we know the α-complex affords a convex covering. The α-complex is then isomorphic to the union covering of balls $B_u(r)$ decomposed by Vornoi cells.

Considering the α-complex in $\mathbb{R}^2$ as shown in 6.8 we see that straight line boundaries form between intersecting balls as a result of the Voronoi decomposition and curved boundaries for non-intersecting cells due to the radius constraints from the union of balls. A common natural example is the structure observed from a collection of soap bubbles [Saye].

Figure 6.8: α-complex and associated nerve.
Image modified from [EdelHarer].



Biographical Notes: Alpha complexes were introduced first for points in $\mathbb{R}^2$ by Edelsbrunner, Kirkpatrick, and Seidel [Edelsetal] and then extended to $\mathbb{R}^3$ in [Edels]

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