CS 6170 Computational Topology: Topological Data Analysis **University of Utah**
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5.1 Simplicial Complexes

5.1.1 Types of Simplicial Complexes

- Simplex
- Simplicial Complex (SC)
- Abstract Simplicial Complex

Typical Simplicial Complexes:

1. Vietoris-Rips (Rips Complex)
2. Cech Complex
3. Delauney Complex (Overlaps with computational geometry and related closely to Vornoi Diagram)
4. Alpha Complex (Used in protein docking ; The company GeoMagic uses Alpha Complex)

Sparsified Simplicial Complexes (that we will study later in the course)

1. Witness Complex
2. Graph Induced Complex

Here is an illustration of an Interactive Voronoi Diagram Generator

5.1.2 Combinatorial Structure Point Cloud Data(PCD)

- Graph
  Describes a ”pairwise” relation between data. An Abstract Graph is a pair G = (V, E) consisting of a set of vertices V , and a set of Edges E , each a pair of vertices.The Graph is Simple if the edge set is a subset of the set of unordered pairs.

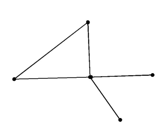

Figure 5.1: A Graph

- Simplicial Complex
  Describes "Higher-Order" interactions (includes the pairwise interactions from a graph) and the Laplacian is well defined. A Simplicial Complex is a finite collection of simplices K such that

$$\sigma \in K \textit{ and } \tau \leq \sigma \textit{ implies } \tau \in K$$

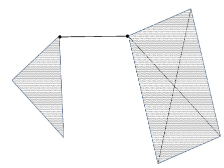

Figure 5.2: A Simplicial Complex

- Hyper-Graph
  Describes an "in-between" structure e.g a hyper-edge among three nodes

5.1.3 Definitions

1. **k-Simplex**
   A k-Simplex is the convex hull of k+1 affinely independent points.
   Suppose $U = \{u_0, \ldots, u_k\}$ is the set of k+1 affinely independent points, then

$$\sigma = \text{Conv}[u_0, ..., u_k]$$

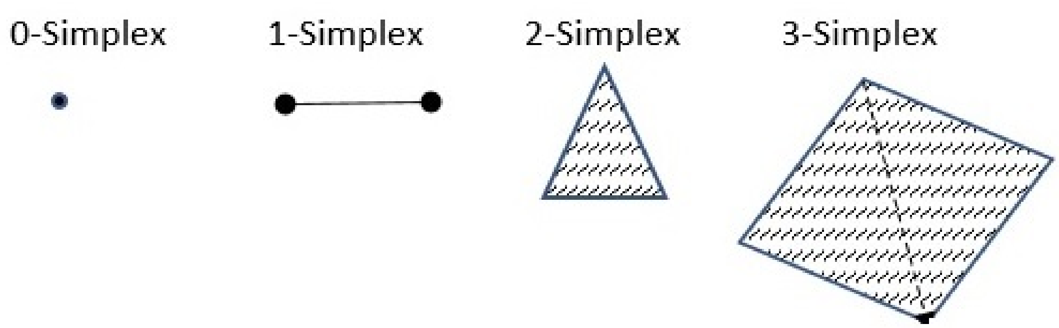

Figure 5.3: k-Simplex

2. **Face**
   A Face of a Simplex σ is the Convex Hull of non-empty subsets of U
   A Face is proper if the subset is not equal to the entire set.

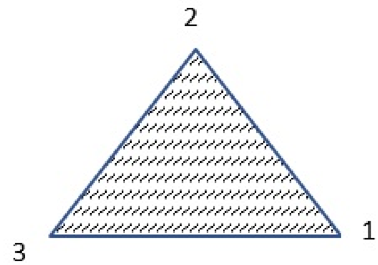

Figure 5.4: 2-Simplex

Face = { 12 , 23 , 13, 1, 2, 3 }
σ = Conv { 1, 2, 3 }
$\tau \leq \sigma$ if τ is a face of σ
$\tau < \sigma$ is a proper face of σ
If τ is a face of σ , σ is a co-face of τ

3. **Boundary**
   The Boundary, τ of a simplex is the Union of all proper faces
   e.g : bd σ = $\bigcup$ { 12 ,23 ,13 ,1 ,2 ,3 }

4. **Abstract Simpicial Complex**
   An Abstract Simplicial Complex is a finite collection of sets A such that
   $\alpha \in$ A , $\beta \subseteq \alpha$, then $\beta \in$ A

5. **Simpicial Complex**
   A Simplicial Complex K is a finite collection of simplices such that
   (i) $\alpha \in$ K , $\tau \leq \sigma \Rightarrow \tau \in$ K
   (ii) σ_1 , $\sigma_2 \in$ K $\Rightarrow \sigma_1 \cap \sigma_2 = \emptyset$, or a face of both

6. **Underlying Space of K**
   | K | the underlying space of K, is the union of Simplices in K together with the topology of the ambient Euclidean Space those simplices live in.

7. **Subcomplex**
   L $\subseteq$ K , a Subcomplex of K is the Simplicial Complex that is a subset of K.

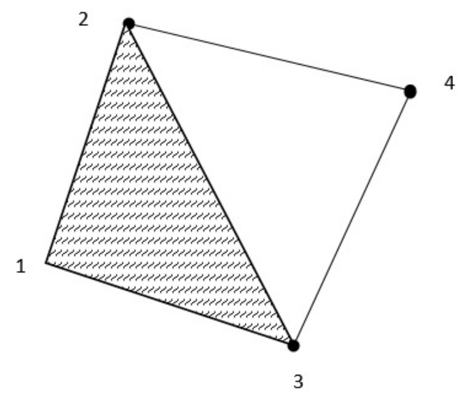

Figure 5.5: Sub Complex

L = { 1, 2 } $\Rightarrow$ Not a Sub Complex
L = { 12, 1, 2 } $\Rightarrow$ Sub Complex

8. **j-Dimensional skeleton**
   A j-Dimensional skeleton of K contains Simplices of Dimension j or less
   K(j) = { $\sigma \in$ K | dim $\sigma \leq$ j }
   K(1) = { 1, 2, 3, 4, 12, 23, 13, 34, 24} $\Rightarrow$ One-Dimensional Skeleton
   K(0) = { 1, 2, 3, 4} = Vertex(K) $\Rightarrow$ 0-Dimensional Skeleton

9. **Local Neighbourhood of a Simplex**

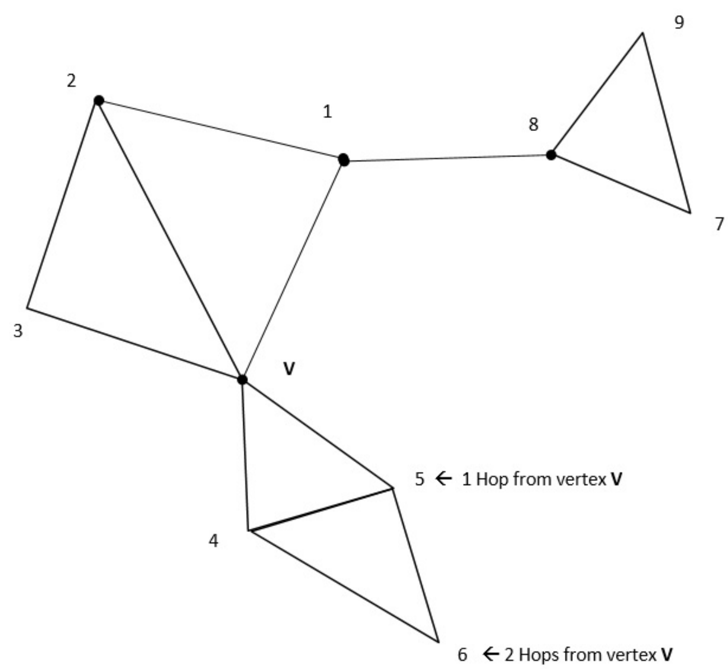

Figure 5.6: Local Neighbourhood of Vertices

Local Neighbourhood of a Simplex, a Star of τ is defined as
St τ = $\{ \sigma \in$ K $\mid \tau \leq \sigma \}$
which implies "all the cofaces of τ"
Example:

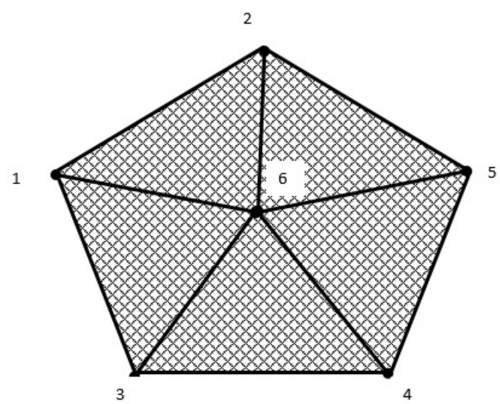

Figure 5.7: Star

Let τ = $\{$ 6 $\}$, then
St τ = $\{$ 16, 26, 36, 46, 56, 126, 256, 564, 346, 136, 6 $\}$

10. **Closed Star**
$\bar{S}t$ the closed star is the smallest sub complex that conatins the star
$\bar{S}t$ τ = St $\tau \cup \{$ 12, 25, 45, 34, 13, 1, 2, 3, 4, 5 $\}$

11. **Link**
A link is a collection of two or more disjoint knots
Link of τ
Lk τ = $\{ \cup \in \bar{S}t$ $\tau \mid$ V $\cap \tau = \emptyset \}$
Example:

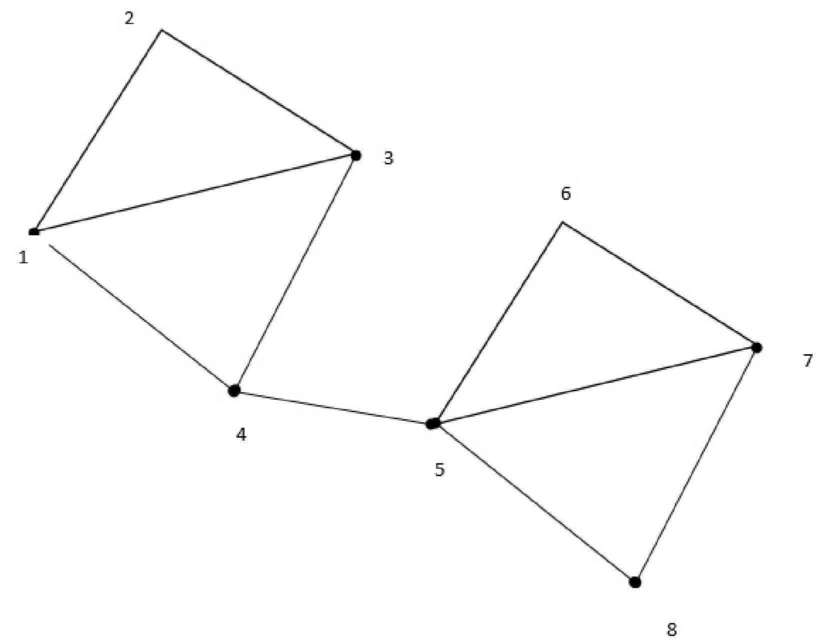

Figure 5.8: Star Example

St (13) = { 123, 134, 13 }

References

[JB00] ALEX BEUTEL , " Interactive Voronoi Diagram Generator" *http://alexbeutel.com/webgl/voronoi.html*,