Persistence-Driven Design and Visualization of Morse Vector Fields (Extended Abstract)

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Abstract

Vector field design on surfaces is originally motivated by applications in graphics such as texture synthesis and rendering. In this extended abstract, we consider the idea of vector field design with a new found motivation in computational topology. We are interested in designing and visualizing vector fields to aid the study of Morse functions in the persistence setting. To achieve such a goal, we develop a new vector field design system that provides fine-grained control over vector field geometry, enables the editing of vector field topology, and supports a design process in a simple and efficient way using elementary building blocks. Our system allows computational topologists to explore the complex configuration space of Morse functions constrained by persistence. Understanding the space of such Morse functions will help us expand the application of persistence for machine learning and visualization.

Index Terms: Morse vector fields, persistence, topological data analysis, vector field design, visualization for mathematics

1 Introduction

The original motivation for vector field design on surfaces originates from diverse applications in graphics (e.g. [6]). A *designer vector field* is used to define texture orientation and scale in example-based texture synthesis, to guide the orientation of brushes and hatches in non-photorealistic rendering, and to simulate fluid flows on smooth surfaces of arbitrary topology. Our work is additionally motivated by advances in computational topology. We would like to design and visualize vector fields to aid the exploration of Morse functions that are considered *equivalent* under the *persistence* setting.

Persistent homology is a powerful tool in topological data analysis that is applicable in both data *summarization* and *simplification*. In its standard setting, persistent homology computes and summarizes topological features of a space $\mathbb{X}$ equipped with a function $f : \mathbb{X} \to \mathbb{R}$ across multiple scales. The importance of a feature can be quantified via the notion of persistence, that is, the amount of change to f necessary to eliminate it. Persistence is also useful in simplifying a function f in terms of removing topological noise as determined by its persistence diagram or barcode [2, 3].

We would like to explore and characterize the set of Morse functions (via their corresponding Gradient vector fields) that give rise to the same barcode. Specifically, we impose an equivalence relation on the space of Morse functions that *respect* persistence, that is, two Morse functions are considered equivalent if they have the same barcode. We further study the dynamics of such Morse functions by exploring the configurations of their gradient vector fields. The combinatorial structures of such an equivalent class can be used to enrich the barcode of a Morse function for machine learning and visualization.

We develop a vector field design system that supports not only visualization and graphics researchers, but also applied and computational topologists. We consider an ideal vector field design system to possess certain desirable traits. Such a system should:

1. Provide control over vector field topology, such as types and locations of singularities, and geometry of the separatrices;
2. Generate a vector field with the exact topology as the user intended;
3. Enable addition, detection and editing of the vector field topology;
4. Support a design process in a simple and efficient way.

In addition, inspired by topological data analysis, our system will encode and embrace the notion of persistence, in particular, it will:

1. Compute and visualize the barcode of a designer vector field to offer a global summary of its features;
2. Support the adjustment of function values at each singularities to explore diverse Morse functions, their gradient vector field configurations and barcodes;
3. Create a one-to-one mapping between the topological features in the domain with bars in the barcode to guide interactive vector field simplification.

2 Technical Background

Vector field topology. A (2D) vector field (flow) v is a smooth mapping $v : \mathbb{M} \to \mathbb{R}^2$ defined on a 2-dimensional manifold $\mathbb{M}$ (surface). In this paper, we only deal with smooth vector fields on a closed two-dimensional sphere $\mathbb{M} := \mathbb{S}^2$. Although this might seem restrictive, the study of Morse vector field on the sphere under the persistence setting is already nontrivial.

Singularities (or *critical points*) are locations where the vector values are zero. A *streamline* is a line that is tangential to the instantaneous velocity direction. A *topological skeleton* of a vector field consists of singularities and separatrices, which are streamlines connecting the singularities which divide the domain into areas of different flow behavior.

Morse-Smale and Morse vector fields. Let TM_p denote the tangent space of $\mathbb{M}$ at p. A vector field v on $\mathbb{M}$ associates a vector $v(p) \in \mathrm{TM}_p$ to each point $p \in \mathbb{M}$. An *integral curve* of v through a point $p \in \mathbb{M}$ is a smooth map $\gamma : I \to \mathbb{M}$ such that $\gamma(0) = p$ and $\gamma'(t) = v(\gamma(t))$ for all $t \in I$. The image of an integral curve is called a *trajectory* [5, page 10]. Two vector fields $v_1 : \mathbb{M}_1 \to \mathbb{R}^2$ and $v_2 : \mathbb{M}_2 \to \mathbb{R}^2$ are considered *topologically trajectory equivalent* if there is a homeomorphism $h : \mathbb{M}_1 \to \mathbb{M}_2$ that transforms the trajectories of the vector field v_1 into the trajectories of the vector field v_2 preserving the orientations of the trajectories [4, Definition 1.1].

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A vector field v is *structurally stable* if the topological behavior of its trajectories is preserved under small perturbations of v (i.e. the perturbed and the initial vector fields are trajectory topologically equivalent) [4, Definition 1.2].

Suppose $\mathbb{M}$ is compact, there exists a *global flow* $\phi : \mathbb{R} \times \mathbb{M} \to \mathbb{M}$ determined by v such that $\phi(0, p) = p$ and $\phi'(t, p) = v(\phi(x, p))$ [5, Proposition 1.3]. For each $t \in \mathbb{R}$, the map $v_t : \mathbb{M} \to \mathbb{M}$ is defined as $v_t(p) = \phi(t, p)$. The ω*-limit set* of a point $p \in \mathbb{M}$, $\omega(p) = \{q \in \mathbb{M} \mid v_{t_n}(p) \to q$ for some sequence $t_n \to \infty\}$. The α*-limit set* of p, $\alpha(p) = \{q \in \mathbb{M} \mid X_{t_n} \to q$ for some sequence $t_n \to -\infty\}$.

A vector field v on a closed two-dimensional surface is called a *Morse-Smale vector field* [4, Definition 1.4] if

- v has finitely many singular points and periodic trajectories, which are all hyperbolic;
- There are no trajectories from a saddle to a saddle;
- The α-limit set and the ω-limit set of each trajectory of v is either a singular point or a periodic trajectory (a limit cycle).

A vector field is a *Morse vector field* if it is a Morse-Small vector field without periodic trajectories.

A vector field is *gradient like* if it is topologically trajectory equivalent to the gradient vector field grad$f$ of a function f and a Riemannian metric g_{ij} on $\mathbb{M}$. Morse vector fields are precisely the gradient-like vector fields without saddle-saddle connections (separatrices from a saddle to a saddle) [7].

3 Vector Field Design

Elementary building blocks. Our method is based on understanding how cells (generically as quadrangles) of a Morse vector field (Fig. 1 top left) can fit together on a surface and how they change when a pair of singularities is added to their interiors and boundaries; such operations are referred to as *elementary building blocks*, which originate from a forthcoming theoretic work [1]. As illustrated in Fig. 1: adding a pair (of singularities) in the interior of a cell is a face move (top row); adding a pair on an edge between two cells is an edge move (middle row); and adding a pair to be colocated with an existing singularity is a vertex move (bottom row).

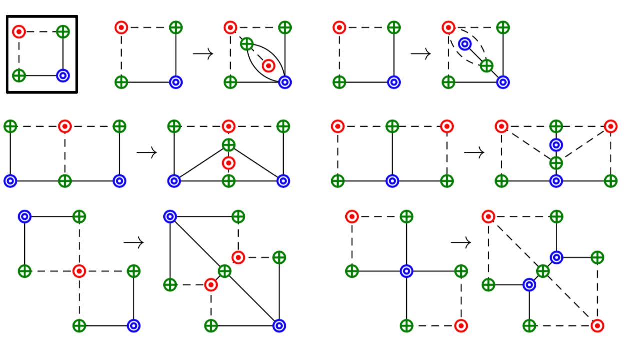

Figure 1: Elementary building blocks. Top left corner: a cell from a Morse vector field. First row: face-max and face-min moves. Second row: edge-max and edge-min moves. Third row: vertex-max, vertex-min moves.

Vector field construction. Once the user specifies the types and locations of singularities via the elementary building blocks, we use the framework of Zhang et al. [8] to construct an initial vector field. We attach a *basis vector field* to each (user-specified) singularity; and a designer vector field is constructed as the truncated sum of these basis vector fields. For instance, a basis vector field centered at a source $\mathbf{p}_0 = (x_0, y_0)$ is defined as: $V(\mathbf{p}) = e^{-d\|\mathbf{p}-\mathbf{p}_0\|^2} \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$, for any point $\mathbf{p} = (x, y) \in \mathbb{R}^2$. For sinks and saddles, we replace the above matrix $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ with $\begin{pmatrix} -k & 0 \\ 0 & -k \end{pmatrix}$ and $\begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}$, respectively.

Geometric control of the separatrices. To provide geometry control of separatrices, we approximate the geometry of the separatrices using cubic cardinal splines. In addition to the initial vector field, we also generate another auxiliary vector field which captures the flow along separatrices using splines. The final vector field is a weighted sum of the initial and the auxiliary vector fields.

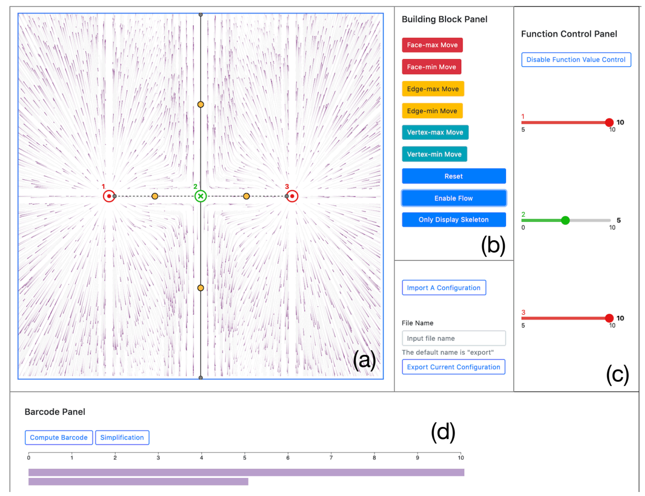

Figure 2: An overview of our vector field design system.

Visualization Design. A layout of our design system is provided in Figure 2, see the supplementary video for a demo. The system contains a **flow visualization panel** (a) that provides capabilities in modifying the geometry of separatrices via control points of the splines; it also visualizes the dynamics of the underlying vector field via animations. The various building blocks are listed on the **building block panel** (b). Under the *manual mode*, a user connects pairs of singularities manually and our system checks for feasible configurations. Using the *semi-automatic mode*, edges are added automatically, followed by user adjustments. The **function control panel** (c) enables a user to modify the function values at singularities. Finally, using the **barcode panel** (d), we provide persistence barcode of the current vector field configuration; we also provide interactive capabilities for persistence-based vector field simplification.

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