

# Math 6610: Analysis of Numerical Methods, I

## Floating-point arithmetic and computer algorithms

Department of Mathematics, University of Utah

Fall 2025

Resources: Trefethen and Bau 1997, Lectures 12-15  
Atkinson 1989, Chapter 1  
Salgado and Wise 2022, Sections 4.2, 4.3

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There's an anatomy to this breakdown:

- The portion 34 is the “integer” part.
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Without context, the **base**  $b$  could be other positive integers:

$$b = 6 \Rightarrow 3 \times 6^1 + 4 \times 6^0 + 1 \times 6^{-1} + 5 \times 6^{-2} + 0 \times 6^{-3} + 3 \times 6^{-4}$$

Note that by convention  $b$  must be strictly larger than all the digits.

(So 34.1503 could be base 6, or 10, or 978, but could not be base 5.)

## Computer representations

Circuits typically detect the presence (1) or absence (0) of an electrical signal.

For this reason, base  $b = 2$  is the standard computer format, e.g.,:

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Q: Why do we always write numbers in base 10?

## Fixed point vs floating point

Fixed-point representations have a fixed radix point, with the size of the fractional part predetermined:

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Then the fractional precision of this representation is fixed for any number.

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Floating-point representations allow the radix to float:

$$\textcolor{blue}{1}.\underbrace{000100010011001111}_{\text{fixed number of entries}} + \textcolor{red}{\text{exponent}}$$

The **exponent** encodes which exponent the radix is aligned with. (Above: 5)

# Floating-point representations

Generally speaking, floating-point representations store:

significand + exponent + sign

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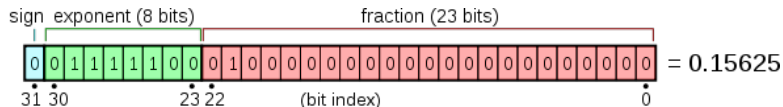
The most popular representation is the IEEE 754 standard, defining various formats:

- Binary 16
- Binary 32 (“single precision”)
- Binary 64 (“double precision”)
- $\vdots$
- Complex numbers are typically stored as two floats, e.g., Complex 128 (64 + 64)

Most high-level scientific computing languages use double precision as default.

# Floating-point details

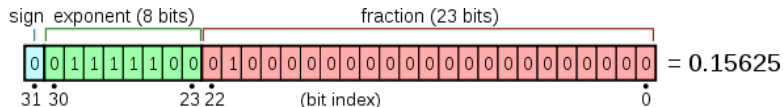
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The number of bits allocated to the exponent indicates the rounding precision of the format.

Machine precision or machine epsilon is the maximum relative rounding error due to finite representation. If  $\text{fl}(x)$  is the floating-point representation of a number  $x$ , then machine precision is the maximum value of

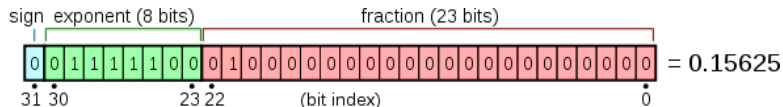
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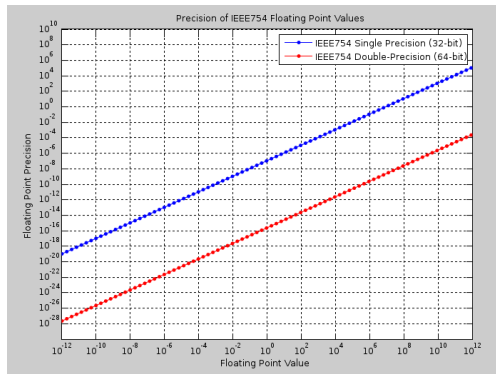
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The roundoff or truncation error associated with a floating-point format is essentially equal to  $\text{base}^{-(\#\text{significant digits})}$ .

# Machine precision

Floating-point representation ensures that there is not really an absolute error committed by computer representations, there is only a relative error.



Source: <https://en.wikipedia.org/wiki/File:IEEE754.svg>

## Roundoff and computing

The rounding truncation imparted by finite representations requires attention to how algorithms are implemented. E.g.,

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- Implementation of the formula  $\sqrt{1+x^4} - 1$  for small positive  $x$ .
- Evaluation of  $e^x$  for  $x < 0$ .
- Evaluation of  $\frac{f(x+h)-f(x)}{h}$  for small  $h$ .

# Solution sensitivity

Goal: understand sensitivity of numerical algorithms to roundoff errors. (“stability”)

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### Example

Let  $f(x) = ax$  for scalars  $a, x$ .

The sensitivity of the map  $x \mapsto f(x)$  depends on the value of  $a$ .

## Absolute sensitivity measures

Let  $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ .

A sensible measure of sensitivity of  $f$  at  $x$  is perturbation-based:

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As with derivatives, we can measure the sensitivity at  $x$  by taking limits:

$$\hat{\kappa}_f(x) := \lim_{\delta \rightarrow 0} \sup_{\|\delta x\| \leq \delta} \frac{\|\delta f\|}{\|\delta x\|}, \quad \delta f := f(x + \delta x) - f(x)$$

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$\hat{\kappa}_f(x)$  is called the absolute condition number of  $f$  at  $x$ .

Note that condition numbers are properties of the map  $f$  and *not* of an algorithmic or finite-precision implementation.

## Relative sensitivity measures

Recall: floating-point arithmetic makes relative errors, not absolute ones.

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The relative condition number of  $\mathbf{f}$  at  $\mathbf{x}$  is defined as

$$\begin{aligned}\kappa_{\mathbf{f}}(\mathbf{x}) &:= \lim_{\delta \rightarrow 0} \sup_{\|\delta \mathbf{x}\| \leq \delta} \frac{\frac{\|\delta \mathbf{f}\|}{\|\mathbf{f}(\mathbf{x})\|}}{\frac{\|\delta \mathbf{x}\|}{\|\mathbf{x}\|}} \\ &= \lim_{\delta \rightarrow 0} \sup_{\|\delta \mathbf{x}\| \leq \delta} \frac{\|\delta \mathbf{f}\| \|\mathbf{x}\|}{\|\delta \mathbf{x}\| \|\mathbf{f}(\mathbf{x})\|}\end{aligned}$$

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Again, this is a (mathematical) property of  $f$ , and not of an algorithmic implementation.

Problems (functions  $f$ ) with “small” condition numbers are *well-conditioned*.

Problems (functions  $f$ ) with “large” condition numbers are *ill-conditioned*.

Just like derivatives and differentials, the condition number is a sensitivity of relative errors:

$$\frac{\|\delta f\|}{\|f(x)\|} \lesssim \kappa_f(x) \frac{\|\delta x\|}{\|x\|}.$$

# Examples

## Example

If  $\mathbf{f} : \mathbb{C}^n \rightarrow \mathbb{C}^m$  is smooth, then condition numbers are norms of Jacobians.

$$\begin{aligned}\hat{\kappa}_{\mathbf{f}}(\mathbf{x}) &= \|\mathbf{J}(\mathbf{x})\|, & \mathbf{J}(\mathbf{x}) &:= \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}), \\ \kappa_{\mathbf{f}}(\mathbf{x}) &= \frac{\|\mathbf{J}(\mathbf{x})\| \|\mathbf{x}\|}{\|\mathbf{f}(\mathbf{x})\|}\end{aligned}$$

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$f : \mathbb{C} \rightarrow \mathbb{C}$  defined by  $f(x) = ax$ .

Compute  $\hat{\kappa}_f(x)$  and  $\kappa_f(x)$ .

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$f : \mathbb{C} \rightarrow \mathbb{C}$  defined by  $f(x) = x^p$  for arbitrary  $p > 0$ .  
Compute  $\hat{\kappa}_f(x)$  and  $\kappa_f(x)$ .



# Examples

## Example

Compute condition numbers for  $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$  defined by  $f(x) = Ax$  for invertible  $A \in \mathbb{C}^{n \times n}$ .

## Linear problems

The (relative) condition number of the linear map  $x \mapsto Ax$ , for invertible  $A$ , is bounded by

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More generally, given invertible  $A$ , the (matrix) condition number of  $A$  is defined as

$$\kappa(A) = \|A\| \|A^{-1}\|.$$

# Numerical algorithms

Given  $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ , we wish to understand how roundoff errors affect evaluation of  $f$ .

This should depend on

- Loss of accuracy due to finite precision
- Conditioning of  $f$

We need condition numbers here since we'll make a small relative mistake in  $x$  (roundoff/truncation), so we must understand what kind of relative mistake we make in  $f(x)$ .

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However, things are somewhat more complicated because our implementation is never exactly  $f$ .

Let  $\tilde{f} : \mathbb{C}^n \rightarrow \mathbb{C}^m$  denote the actual algorithmic implementation of  $f$ .

For example, the numerical implementation of  $f(x) = 1 + x$  might be  $\tilde{f}(x) = \text{fl}(1) \oplus \text{fl}(x)$ , where  $\oplus$  is some implementation of the addition of two floating point numbers.

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We might hope that

$$\frac{\|f(x) - \tilde{f}(x)\|}{\|f(x)\|} = \mathcal{O}(\epsilon_{\text{mach}}),$$

and one suspects that this condition might be tied to conditioning of  $f$ .

## Forward error

Suppose  $f$  is the “exact” procedure, and  $\tilde{f}$  is the “approximate” (computational) procedure.

The **forward error** at  $x$  is the relative error committed by  $\tilde{f}$ :

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This notion of error is typically used to understand stability as follows:

$$\text{FE}(x) \leq \frac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(x)\|} + \frac{\|f(\tilde{x}) - f(x)\|}{\|f(x)\|},$$

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where we’ve introduced an intermediate quantity  $\tilde{x}$ . Ideally, this quantity is a “small” perturbation of  $x$ . In particular, if  $\tilde{x}$  is “close” to  $x$ , then the latter term is

$$\frac{\|f(\tilde{x}) - f(x)\|}{\|f(x)\|} \lesssim \kappa(x) \frac{\|x - \tilde{x}\|}{\|x\|}.$$

All of this motivates a requirement on the behavior first term, which is purely an algorithmic concern.

## Forward stability

### Definition

An algorithm  $\tilde{f}$  is *forward stable* at  $x$  if there is some  $\epsilon > 0$  so that,

$$\frac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(x)\|} \lesssim \kappa(x) \frac{\|x - \tilde{x}\|}{\|x\|},$$

whenever  $\|x - \tilde{x}\|/\|x\| \leq \epsilon$ .

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If an algorithm is forward stable and the procedure is well-conditioned, then we have our desired error:

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This required error estimation procedure, establishment of forward stability of  $f$ , is *forward error analysis*. Showing forward stability is frequently technical and difficult.

## Backward stability

A dual, but somewhat stranger+stronger notion of stability is backward stability.

From this point of view, we essentially assume that the algorithm produces “exactly the right answer to a closely related question”.

### Definition

An algorithm  $\tilde{f}$  is backward stable at  $x$  if we have

$$\tilde{f}(x) = f(\tilde{x}), \quad (1)$$

for some  $\tilde{x}$  satisfying  $\|x - \tilde{x}\| = \|x\|\mathcal{O}(\epsilon)$ .

(Again, to make this practical, one requires  $\epsilon = \epsilon_{\text{mach}}$ .)

An ancillary definition: the corresponding notion of backward error of  $\tilde{f}$  at  $x$  is,

$$\text{BE}(x) = \inf \left\{ \frac{\|x - \tilde{x}\|}{\|x\|} \mid \tilde{f}(x) = f(\tilde{x}) \right\}.$$

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If  $\tilde{f}$  is backward stable, then showing accuracy of the algorithm  $\tilde{f}$  is easier.

Estimating error by establishing backward stability is *backward error analysis*. Backward stability is frequently easier to show than forward stability, but forward stable algorithms need not be backward stable.

## Examples

Assume some floating-point axioms:

1. For each  $x \in \mathbb{R}$ , then  $fl(x) = x(1 + \epsilon)$  for  $|\epsilon| \leq \mathcal{O}(\epsilon_{\text{mach}})$ .
2. For floating-points numbers  $x, y \in \mathbb{R}$ , then  $x \odot y = (x \cdot y)(1 + \epsilon)$  for  $|\epsilon| \leq \mathcal{O}(\epsilon_{\text{mach}})$ . ( $\cdot = +, -, \times$ )



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## Examples

Assume some floating-point axioms:

1. For each  $x \in \mathbb{R}$ , then  $fl(x) = x(1 + \epsilon)$  for  $|\epsilon| \leq \mathcal{O}(\epsilon_{\text{mach}})$ .
2. For floating-points numbers  $x, y \in \mathbb{R}$ , then  $x \odot y = (x \cdot y)(1 + \epsilon)$  for  $|\epsilon| \leq \mathcal{O}(\epsilon_{\text{mach}})$ . ( $\cdot = +, -, \times$ )

### Example

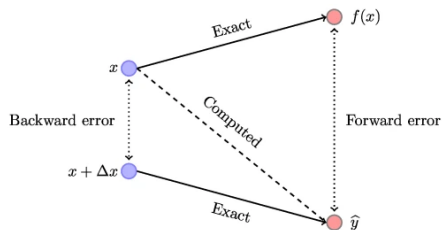
The floating-point implementation of  $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{x}\mathbf{y}^T$  is *not* backward stable.

## Some punch lines

*Conditioning* is a property of mathematical maps/functions.

*Stability* is a property of an algorithmic implementation.

Well-conditioned functions *and* stable implementations yield reliable accuracy.



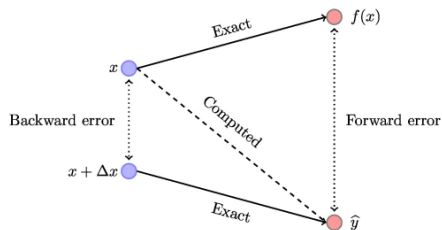
Source: <https://nhigham.com/2020/03/25/what-is-backward-error>

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Studying conditioning of maps is essential to understanding best-possible behavior of algorithms. An ill-conditioned operation with a stable implementation wouldn't yield reliably accurate results.

## Eigenvalue conditioning

Weren't we interested in eigenvalues....?

Consider  $\mathbf{A} \in \mathbb{C}^{n \times n}$ , along with the map  $\lambda : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}$  that computes an eigenvalue of  $A$ :

$$\mathbf{A}\mathbf{v} = \lambda(\mathbf{A})\mathbf{v}, \quad \text{for some } \mathbf{v} \neq \mathbf{0}$$

What is the conditioning of this operation?

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### Theorem (Bauer-Fike)

*Assume  $\mathbf{A} \in \mathbb{C}^{n \times n}$  is diagonalizable with eigenvector matrix  $\mathbf{V} \in \mathbb{C}^{n \times n}$ . Then, using the 2-norm, the absolute condition number of computing eigenvalues is  $\tilde{\kappa}_\lambda(\mathbf{A}) = \kappa(\mathbf{V})$ .*



## References I

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