Math 6610: Analysis of Numerical Methods, I
Eigenvalues and eigenvectors

Department of Mathematics, University of Utah

Fall 2025

Resources: Trefethen and Bau 1997, Lecture 24
Süli and Mayers 2003, Section 5.1
Salgado and Wise 2022, Sections 1.3, 8.3

Eigenvalues and eigenvectors

Given $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, $(\lambda, \boldsymbol{v}) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{0\})$ is an eigenvalue-eigenvector pair if

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}.$$

Recall: it doesn't matter what value(s) λ takes on, but $\boldsymbol{v}$ *cannot* be $\mathbf{0}$.

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$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Some additional terminology/properties:

- The collection of all eigenvalues of $\boldsymbol{A}$ is $\lambda(\boldsymbol{A}) \subset \mathbb{C}$, its *spectrum*.
- Even if $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\lambda(\boldsymbol{A})$ can contain complex values, and eigenvectors can be complex-valued.
- On paper, we typically identify the spectrum by computing roots of the *characteristic polynomial*, $p_{\boldsymbol{A}}(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A})$. (This turns out to be a *terrible* algorithmic strategy.)

Eigenpair properties

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- All square matrices have exactly n eigenvalues, with some possibly repeated.
- All square matrices have at least 1 eigenvector.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

(Eigenvectors aren't unique,
because of scaling)

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- Eigenvectors from *distinct* eigenvalues are linearly independent.
- Every *distinct* eigenvalue has at least 1 eigenvector.

why?

$A v_1 = \lambda_1 v_1$

$A v_2 = \lambda_2 v_2$ $\quad \lambda_1 \neq \lambda_2$

Consider $c_1 v_1 + c_2 v_2$ (arbitrary c_1, c_2)

$c_1 v_1 + c_2 v_2 \overset{?}{=} 0$

$$\left.\begin{aligned} c_1 \lambda_1 v_1 + c_2 \lambda_1 v_2 &\overset{?}{=} 0 \\ c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 &= 0 \end{aligned}\right\} \Longrightarrow c_1 = c_2 = 0$$

A

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- Every *distinct* eigenvalue has at least 1 eigenvector.
- The collection of all eigenvectors associated to an eigenvalue λ is a subspace, and is frequently called an *eigenspace* E_λ.
- Eigenspaces are invariant subspaces of $\boldsymbol{A}$, i.e., $\boldsymbol{A}E_\lambda \subseteq E_\lambda$.

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad E_{\underline{1}} = \text{span}\{\underline{e}_1, \underline{e}_2\}$$

$E = \mathbb{C}^3$ is an invariant subspace of $\underline{\underline{A}}$

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- The number of times an eigenvalue is repeated a_λ is its *algebraic multiplicity*
- The *geometric multiplicity* g_λ of an eigenvalue λ is $\dim E_\lambda$.
- *Simple* eigenvalues λ have $g_\lambda = a_\lambda = 1$.

$$\sum_{\lambda \in \lambda(A)} a_\lambda = n$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \lambda(A) = \{1\}$$

$$a_1 = 2, \quad g_1 = 1$$

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- Eigenvalues λ with $g_\lambda < a_\lambda$ are *defective*
- Any $\boldsymbol{A}$ with a defective eigenvalue is *defective*.

$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\lambda(\underline{A}) = \{1\}$

$a_1 = 3, \quad g_1 = 2$

A digression: similarity transforms

Two square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are *similar* if $\exists$ an invertible $\boldsymbol{S}$ such that

$$\boldsymbol{B} = \boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}.$$

(The map $\boldsymbol{A} \mapsto \boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}$ is a similarity transform.)

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Suppose $\boldsymbol{A}$ is not defective: then with linearly independent $\boldsymbol{v}_i$, $i \in [n]$ we have the relations,

$$\boldsymbol{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i.$$

This is equivalent to,

$$\boldsymbol{A}\boldsymbol{V} = \cancel{\boldsymbol{\Lambda}\boldsymbol{V}}, \qquad \boldsymbol{V} = \begin{pmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{pmatrix}, \qquad \boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

$\underline{V}\,\underline{\Lambda}$

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And since $\boldsymbol{V}$ is full-rank, then

$$\boldsymbol{A} = \cancel{\boldsymbol{V}^{-1}\boldsymbol{\Lambda}\boldsymbol{V}}, \quad \underline{V}\,\underline{\Lambda}\,\underline{V}^{-1}$$

i.e., $\boldsymbol{A}$ is similar to a diagonal matrix containing its spectrum.

Matrix diagonalizability

This motivates a key definition and consequence.

Definition

A square matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ is <u>diagonalizable</u> if it is similar to a diagonal matrix.

Theorem

A square matrix $\boldsymbol{A}$ is diagonalizable iff it is not defective.

When $\boldsymbol{A}$ is not defective, it is diagonalizable via a matrix whose columns are comprised of its linearly independent eigenvectors.

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Similarity invariances

One simple observation is that the set of eigenvalues is invariant under a(ny) similarity transform, since if $\boldsymbol{A}$ is diagonalizable with diagonal matrix $\boldsymbol{\Lambda}$, then,

$$\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = (\boldsymbol{V}\boldsymbol{S})^{-1}\boldsymbol{\Lambda}(\boldsymbol{V}\boldsymbol{S})$$

is a diagonalization of $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}$ with the same diagonal matrix $\boldsymbol{\Lambda}$.

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More consequences follow, e.g., some familiar determinant and trace properties,

$$\det \boldsymbol{A} = \prod_{j=1}^{n} \lambda_j, \qquad \mathrm{tr}\boldsymbol{A} = \sum_{j=1}^{n} \lambda_j.$$

The above is actually true for any square matrix $\boldsymbol{A}$, defective or not.

Jordan normal form

Defective matrices certainly exist. The most common example is,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

A rather nice fact is that the above example is "spiritually" the prototypical and essentially only type of defective matrix.

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Theorem (Jordan normal form)

Every square matrix $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ is bidiagonalizable, i.e., is similar to a bidiagonal matrix.

More specifically, let $\lambda_1, \ldots, \lambda_U$ be the unique eigenvalues of $\boldsymbol{A}$, and suppose they have algebraic and geometric multiplicities a_j and g_j, $j \in [U]$, respectively. Tnen:

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1}, \qquad \boldsymbol{V} \in \mathbb{C}^{n\times n},$$

where $\boldsymbol{V}$ is invertible, and $\boldsymbol{J}$ is bidiagonal with $\Lambda(\boldsymbol{A})$ on the diagonal, given by,

$$\boldsymbol{J} = \bigoplus_{j\in[U]} \left(\lambda_j \boldsymbol{I}_{g_j-1} \oplus \boldsymbol{J}_j\right), \qquad \boldsymbol{J}_j = \lambda_j \boldsymbol{I}_{a_j-g_j+1} + \boldsymbol{N}_{a_j-g_j+1},$$

where $\boldsymbol{N}_k$ is a $k \times k$ matrix, nonzero only on its main superdiagonal that has entries all 1.

Given $\underline{\underline{A}}$, spectrum $\lambda_1, \lambda_2, \lambda_3$

$$\begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda_3 \\ \uparrow & \uparrow & \uparrow \\ a=1 & a=2 & a=2 \\ g=1 & g=2 & g=1 \end{array}$$

$\exists\ \underline{\underline{V}}$ invertible such that

$$\underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}} = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_2 & & \\ & & & \lambda_3 & 1 \\ & & & & \lambda_3 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 \underline{\underline{I}}_1 & & \\ & \lambda_2 \underline{\underline{I}}_2 & \\ & & \lambda_3 \underline{\underline{I}}_2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

$$= \lambda_1 \underline{\underline{I}}_1 \oplus \lambda_2 \underline{\underline{I}}_2 \oplus \left[\lambda_3 \underline{\underline{I}}_2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right]$$

To diagonalizability and beyond

"Most" square matrices are diagonalizable.

This is (incredibly) powerful: a symmetric linear change of basis of the input and output spaces results in a diagonal linear operator.
The particular change of basis can be quite anisotropic (and non-isometric) in nature.

All square matrices are not diagonalizable. However, ...

$$\underline{\underline{V}} = \begin{pmatrix} 1 & 1 \\ 1 & 1+\varepsilon \end{pmatrix}$$

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Are there matrices that are *unitarily* diagonalizable?

A "spectral" theorem

A seemingly unrelated algebraic definition is our starting point.

Definition

A matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ is *Hermitian* if $\boldsymbol{A} = \boldsymbol{A}^*$.
(Hermitian matrices are also called *self-adjoint*, or *symmetric* when $\boldsymbol{A}$ is real-valued.)

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Theorem (Spectral Theorem for Hermitian matrices)

If $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ is Hermitian, then it is unitarily diagonalizable with real eigenvalues.
(Its spectrum is real-valued, and the similarity matrix accomplishing diagonalization is unitary.)

Proof: (i) Show that $\lambda(\underline{A})$ is real.

Let $(\lambda, \underline{v})$ be any eigenpair with $\|\underline{v}\|_2^2 = 1$

$$\lambda \cdot 1 = \lambda \|\underline{v}\|_2^2 = \lambda \langle \underline{v}, \underline{v} \rangle = \langle \lambda \underline{v}, \underline{v} \rangle = \langle \underline{A}\underline{v}, \underline{v} \rangle = \langle \underline{v}, \underline{A}\underline{v} \rangle = \langle v, \lambda v \rangle$$

$= \lambda^* \langle v, v \rangle = \lambda^* \cdot 1$

$\lambda = \lambda^* \Rightarrow \lambda$ real ✓.

(ii) Show that there's a unitary similarity transform that diagonalizes $\underline{\underline{A}}$.

Idea: Let $(\lambda, \underline{v})$ be any eigenpair of $\underline{\underline{A}}$.

Define $V = \operatorname{span}\{\underline{v}\}$.

Then: $V^\perp$ is an invariant subspace of $\underline{\underline{A}}$.

$(\underline{\underline{A}} V^\perp \subseteq V^\perp)$

With this: Can consider $\underline{\underline{A}}$ "restricted" to $V^\perp$, which is an $(n-1)\times(n-1)$ Hermitian matrix.

Iterate...

<u>Details</u>: Let $(\lambda_1, \underline{v}_1)$ be an eigenpair of $\underline{\underline{A}}$

Let $\underline{\underline{Q}}_1$ be unitary, $\underline{\underline{Q}}_1 = \begin{pmatrix} | & | & & | \\ \underline{v}_1 & \underline{q}_2 & \cdots & \underline{q}_n \\ | & | & & | \end{pmatrix}$

($\underline{q}_2, \ldots \underline{q}_n$ are any ON completion of $\mathbb{C}^n$).

$(\|\underline{v}_1\|_2 = 1)$

Consider $Q_1^* A Q_1$: this is Hermitian

$$Q_1^* A Q_1 = Q_1^* \begin{pmatrix} Av_1 & Aq_2 & \cdots & Aq_n \end{pmatrix}$$

$$= Q_1^* \begin{pmatrix} \lambda_1 v_1 & Aq_2 & \cdots & Aq_n \end{pmatrix}$$

$$= \begin{pmatrix} \langle \lambda_1 v_1, v_1 \rangle & \langle Aq_2, v_1 \rangle & \cdots & \langle Aq_n, v_1 \rangle \\ \langle \lambda_1 v_1, q_2 \rangle & \vdots & & \vdots \\ \vdots & & & \\ \langle \lambda_1 v_1, q_n \rangle & \langle Aq_2, v_n \rangle & \cdots & \langle Aq_n, q_n \rangle \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & & & \\ 0 & & \underline{A}_{n-1} & \\ \vdots & & & \\ 0 & & & \end{pmatrix}, \quad \underline{A}_{n-1} \text{ is Hermitian}$$

Iterate: Choose $(\lambda_2, \underline{v}_2)$ as an eigenpair of $\underline{A}_{n-1}$

Define $U_2 = \begin{pmatrix} | & | & & | \\ v_2 & q_2 & \cdots & q_n \\ | & | & & | \end{pmatrix} \in \mathbb{C}^{(n-1)\times(n-1)}$

as a unitary matrix

Define $Q_2 = \begin{pmatrix} I_1 & 0 \\ 0 & U_2 \end{pmatrix} = I_1 \oplus U_2$

Consider $Q_2^* Q_1^* A Q_1 Q_2$

$$= \begin{pmatrix} I_1 & 0 \\ 0 & U_2^* \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & A_{n-1} \end{pmatrix} \cdot \begin{pmatrix} I_1 & 0 \\ 0 & U_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & U_2^* A_{n-1} U_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & A_{n-2} \end{pmatrix}$$

Repeat: $\underbrace{\left(\prod_{j=1}^{n-1} Q_j\right)^*} A \underbrace{\left(\prod_{j=1}^{n-1} Q_j\right)} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$

$$Q = \prod_{j=1}^{n-1} Q_j$$

A "spectral" theorem

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(Hermitian matrices are also called *self-adjoint*, or *symmetric* when $\boldsymbol{A}$ is real-valued.)

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(Its spectrum is real-valued, and the similarity matrix accomplishing diagonalization is unitary.)

Hermitian matrices are *very* common in applications, and the spectral theorem has numerous uses.

Application I: Geometric interpretation of Hermitian matrices

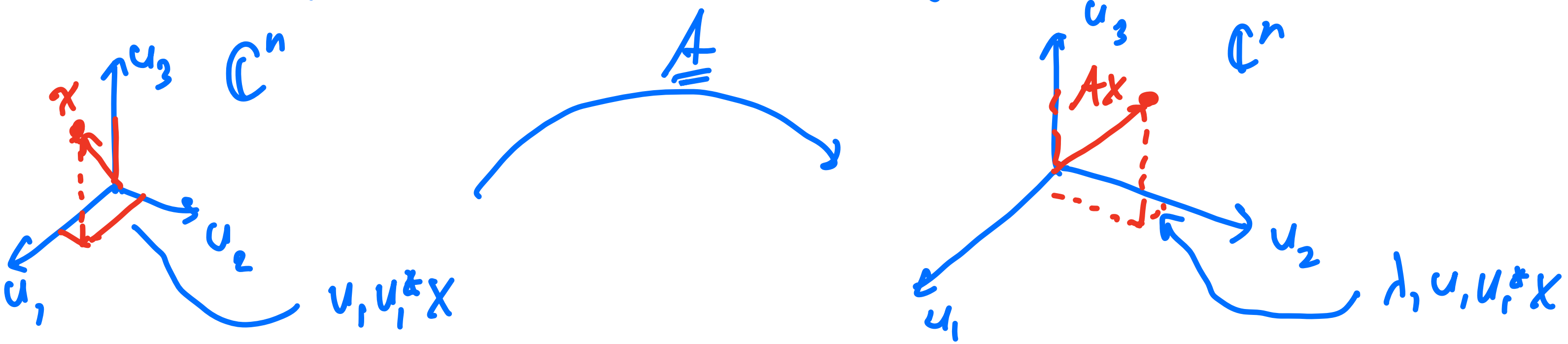
If $A \in \mathbb{C}^{n \times n}$ is unitarily diagonalizable, then it can be written as

$$A = \underbrace{U \Lambda U^*}_{} = \sum_{j=1}^{n} \lambda_j u_j u_j^*,$$

where $\{u_j\}_{j=1}^n$ are the columns of U.

$$\begin{pmatrix} \lambda_1 \underline{u}_1 & \cdots & \lambda_n \underline{u}_n \end{pmatrix} \begin{pmatrix} - & u_1^* & - \\ & \vdots & \\ - & u_n^* & - \end{pmatrix}$$

Recall: $u_j u_j^* x$ is the "vector projection" of x onto u_j



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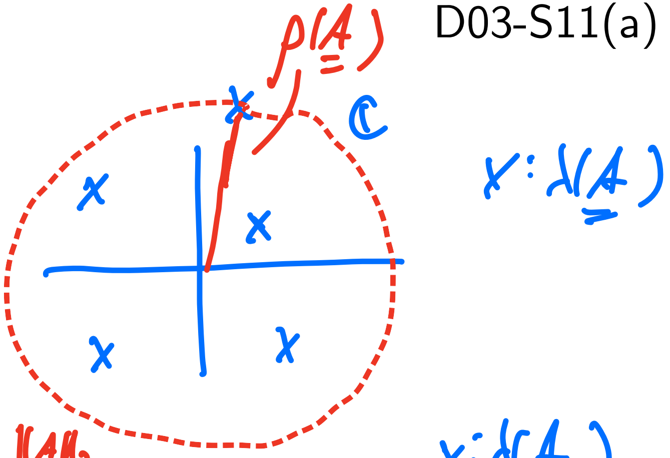
where $\{\boldsymbol{u}_j\}_{j=1}^{n}$ are the columns of $\boldsymbol{U}$.

I.e., Hermitian matrices (an algebraic property) have strong geometric interpretation: they are "just" diagonal matrices in a rotated/reflected orthonormal frame.

Application II: The induced 2-norm



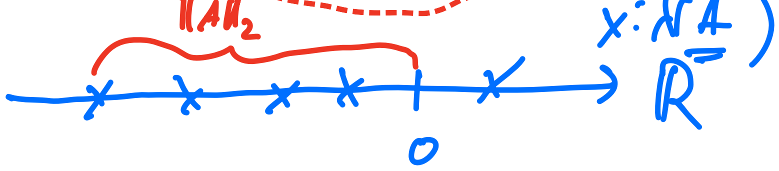
The *spectral radius* of a matrix $\boldsymbol{A}$ is

(cf. $\underline{A} = \begin{pmatrix} 0 & 10^{10} \\ 0 & 0 \end{pmatrix}$)

$$\rho(\boldsymbol{A}) := \max_{j=1,\ldots,n} |\lambda_j(\boldsymbol{A})|$$

If $\boldsymbol{A}$ is Hermitian, then $\|\boldsymbol{A}\|_2 = \rho(\boldsymbol{A})$.

This is direct from the definition of the induced 2-norm.



why? Let $\underline{x}$ satisfy $\|\underline{x}\|_2 = 1$

$$\|\underline{A}\,\underline{x}\|_2 = \Big\| \sum_{j=1}^{n} \underline{u}_j \lambda_j c_j \Big\|_2, \quad c_j = \langle x, u_j \rangle$$

$$= \Big(\sum_{j \in [n]} |c_j \lambda_j|^2 \Big)^{1/2} \leq \rho(\underline{A}) \Big(\sum_{j \in [n]} |c_j|^2 \Big)^{1/2} = \rho(\underline{A})$$

Achievable by choosing x as an eigenvector corresponding to eigenvalue achieves spectral radius

Application III: The "$\boldsymbol{A}$ norm"

A matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ is Hermitian positive definite (sometimes *symmetric* positive-definite or "spd") if it's Hermitian and its (real) spectrum is strictly positive. $\lambda_1, \lambda_2 .. \lambda_n > 0$

(Respectively, positive semi-definite if the spectrum is non-negative.)

(For spd matrices, it's convention to order the spectrum $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n$.)

Ex: $\underline{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is positive semi-definite, not positive definite

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Such matrices actually define a norm: $\|\boldsymbol{x}\|_{\boldsymbol{A}}^2 := \boldsymbol{x}^* \boldsymbol{A} \boldsymbol{x}$ is a norm.

$\underline{x} \mapsto \underline{x}^* \underline{\underline{A}}\, \underline{x}$ is a norm (iff $\underline{\underline{A}}$ is positive definite)

Application IV: Matrix square roots

There is a functional calculus on spd matrices.

For example, a matrix $\boldsymbol{B}$ is the square root of a matrix $\boldsymbol{A}$ if $\boldsymbol{A} = \boldsymbol{B}^2$.

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Example

If $\boldsymbol{A}$ is spd, compute a matrix square root of $\boldsymbol{A}$.

Idea: $\underline{\underline{A}} = \underline{\underline{U}}\,\underline{\underline{\Lambda}}\,\underline{\underline{U}}^*$, $\underline{\underline{\Lambda}}$ has positive diagonal entries

Define $\underline{\underline{B}} = \underline{\underline{U}}\sqrt{\underline{\underline{\Lambda}}}\,\underline{\underline{U}}^*$. Then: $\underline{\underline{B}}^2 = \underline{\underline{U}}\,\underline{\underline{\Lambda}}\,\underline{\underline{U}}^* = \underline{\underline{A}}$

elementwise square root

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Example

If $\boldsymbol{A}$ is spd, compute a matrix square root of $\boldsymbol{A}$.

Theorem

If $\boldsymbol{A}$ is spd, then there is a unique spd square root $\boldsymbol{B}$ of $\boldsymbol{A}$, i.e., $\boldsymbol{B}^2 = \boldsymbol{A}$.

Application IV: Quadratic forms

Given a Hermitian matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$, the function,

$$Q_{\boldsymbol{A}}(\boldsymbol{x}) := \boldsymbol{x}^* \boldsymbol{A} \boldsymbol{x}, = \langle Ax, x \rangle$$

is a **quadratic form**, i.e., a real-valued quadratic function on $\mathbb{C}^n$. The eigendecomposition of $\boldsymbol{A}$ uniquely defines the behavior of $Q_{\boldsymbol{A}}$.

$$Q_A(x) = (U^* x)^* \Lambda (U^* x)$$

$$y = U^* x \qquad = y^* \Lambda y = \lambda_1 |y_1|^2 + \lambda_2 |y_2|^2 + \ldots$$

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is a **quadratic form**, i.e., a real-valued quadratic function on $\mathbb{C}^n$. The eigendecomposition of $\boldsymbol{A}$ uniquely defines the behavior of $Q_{\boldsymbol{A}}$.

If the following eigenvectors correspond to the positive, negative, and zero eigenvalues of $\boldsymbol{A}$, respectively,

$$\left\{\boldsymbol{v}_i^+\right\}_{i \in [n^+]}, \qquad \left\{\boldsymbol{v}_i^-\right\}_{i \in [n^-]}, \qquad \left\{\boldsymbol{v}_i^0\right\}_{i \in [n^0]},$$

where $n = n^+ + n^- + n^0$. Then clearly:

$$Q_{\boldsymbol{A}}(\boldsymbol{v}_i^+) > 0, \qquad Q_{\boldsymbol{A}}(\boldsymbol{v}_i^-) < 0, \qquad Q_{\boldsymbol{A}}(\boldsymbol{v}_i^0) = 0.$$

Application IV: Quadratic forms

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Generalizing this a bit:

spa

$$\left.\begin{array}{l} V^+ := \{\boldsymbol{v}_i^+\}_{i \in [n^+]}, \\ V^- := \{\boldsymbol{v}_i^-\}_{i \in [n^-]}, \\ V^0 := \{\boldsymbol{v}_i^0\}_{i \in [n^0]} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} Q_{\boldsymbol{A}}(\boldsymbol{x}) > 0 \text{ if } \boldsymbol{x} \in V^+ \backslash \{\boldsymbol{0}\} \\ Q_{\boldsymbol{A}}(\boldsymbol{x}) < 0 \text{ if } \boldsymbol{x} \in V^- \backslash \{\boldsymbol{0}\} \\ Q_{\boldsymbol{A}}(\boldsymbol{x}) = 0 \text{ if } \boldsymbol{x} \in V^0 \end{array}\right.$$

where $\mathbb{C}^n = V^+ \oplus V^- \oplus V^0$.

Rayleigh quotients, I

A final application of Hermitian matrices is a *variational* characterization of eigenvalues. We need some buildup for this.

Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ be a(ny) square matrix, and let $\boldsymbol{x} \in \mathbb{C}^n \backslash \{\boldsymbol{0}\}$ be a vector.

The Rayleigh Quotient (of $\boldsymbol{A}$ at $\boldsymbol{x}$) is the (complex) scalar,

$$R_{\boldsymbol{A}}(\boldsymbol{x}) := \frac{Q_{\boldsymbol{A}}(\boldsymbol{x})}{\|\boldsymbol{x}\|_2^2} = \frac{\boldsymbol{x}^* \boldsymbol{A} \boldsymbol{x}}{\boldsymbol{x}^* \boldsymbol{x}} = \frac{\langle \boldsymbol{A}\boldsymbol{x}, \boldsymbol{x} \rangle}{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}, \qquad \boldsymbol{x} \neq \boldsymbol{0}$$

Ostensibly, if $(\lambda, \boldsymbol{v})$ is an eigenpair of $\boldsymbol{A}$, then $R_{\boldsymbol{A}}(\boldsymbol{v}) = \lambda$.

$R_A(\underline{x})$ well-defined for <u>any</u> square matrix.

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$x : N(A)$

$\mathbb{C}$

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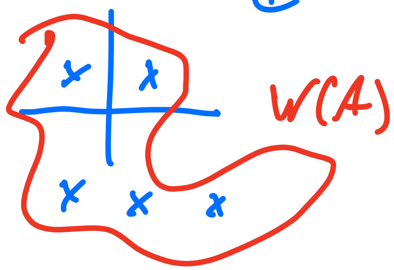


The numerical range of $\boldsymbol{A}$ is the set of all possible values of $R_{\boldsymbol{A}}$:

$$W(\boldsymbol{A}) := R_{\boldsymbol{A}} \left(\mathbb{C}^n \backslash \{\boldsymbol{0}\} \right).$$

$W(A) \subset \mathbb{C}$

One can view $W(\boldsymbol{A})$ as the image of the Rayleigh quotient over all $\mathbb{C}^n$, but also just as the image of the Rayleigh quotient over the unit sphere in $\mathbb{C}^n$.

$W(\boldsymbol{A})$ is some set in $\mathbb{C}$, regardless of the dimension n of $\boldsymbol{A}$.

Clearly we know $\lambda(\boldsymbol{A}) \subset W(\boldsymbol{A})$.

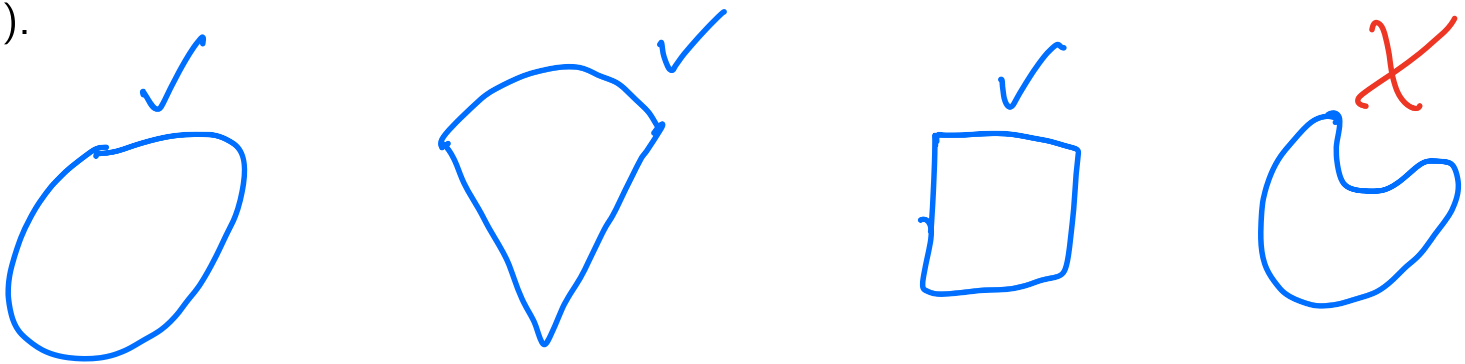
Rayleigh quotients, II

There is a rather more interesting property of the numerical range.

Theorem (Hausdorff-Toeplitz Theorem)

$W(\boldsymbol{A})$ is a compact and convex set in $\mathbb{C}$.

Compactness: $W(\boldsymbol{A})$ is the image of a compact set (unit sphere in $\mathbb{C}^n$) under a continuous function ($R_{\boldsymbol{A}}(\cdot)$).



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For certain classes of matrices, the Rayleigh quotient is a little more transparent.

For example, if $\boldsymbol{A}$ is Hermitian, then $R_{\boldsymbol{A}}(\boldsymbol{x}) \in \mathbb{R}$, so $W(\boldsymbol{A}) \subset \mathbb{R}$.

$\Downarrow$

$W(A) = [a, b] \subset \mathbb{R}$, $|a|, |b| < \infty$.

Suppose $\underline{x}$ is unit-norm: $\|x\|_2 = 1$, $R_{\underline{A}}(\underline{x}) = x^* A x = x^* U^* \Lambda U x$

Define $\underline{c} = Ux \Rightarrow R_A(x) = c^* \Lambda c = \sum_{j \in [n]} \lambda_j |c_j|^2$, $\lambda_j \in \mathbb{R}$

Order λ_j: $\lambda_1 \le \lambda_2 \le \dots \le \lambda_n$

$$\Rightarrow R_A(\underline{x}) = \sum_{j \in [n]} \lambda_j |c_j|^2 \le \sum_{j \in [n]} \lambda_n |c_j|^2 = \lambda_n \|c\|_2^2 = \lambda_n$$

Similarly: $R_A(x) \ge \lambda_1$

These are achievable: $A v_1 = \lambda_1 v_1 \Rightarrow R_A(v_1) = \lambda_1$

$A v_n = \lambda_n v_n \Rightarrow R_A(v_n) = \lambda_n$

Rayleigh quotients, II

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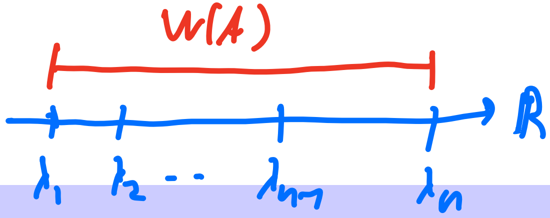

In fact, something more precise is true

Theorem

If $\boldsymbol{A}$ is Hermitian, then

$$\lambda_{\min}(\boldsymbol{A}) \leqslant R_{\boldsymbol{A}}(\boldsymbol{x}) \leqslant \lambda_{\max}(\boldsymbol{A}), \qquad \boldsymbol{x} \in \mathbb{C}^n \backslash \{\boldsymbol{0}\}.$$

An immediate corollary: If $\boldsymbol{A}$ is Hermitian, then $W(\boldsymbol{A}) = [\lambda_{\min}(\boldsymbol{A}), \lambda_{\max}(\boldsymbol{A})]$.

Rayleigh quotients, III

Let $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ be Hermitian. Consider a subspace $V \subset \mathbb{C}^n$.

The image of the V under the Rayleigh quotient, $R_{\boldsymbol{A}}(V)$, is some subset of $W(\boldsymbol{A}) \subset \mathbb{R}$.

$$\shortparallel$$

$$\{R_A(x) \mid x \in V \setminus \{0\}\}$$

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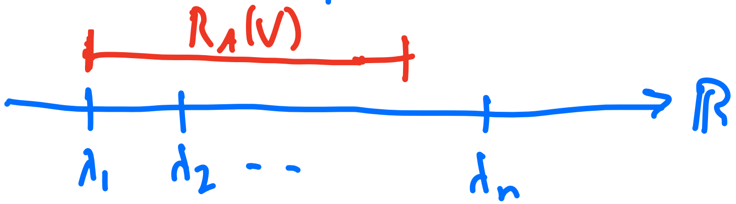
The image of the V under the Rayleigh quotient, $R_{\boldsymbol{A}}(V)$, is some subset of $W(\boldsymbol{A}) \subset \mathbb{R}$.

- The minimum value of $R_{\boldsymbol{A}}(V)$ is $\lambda_{\min}(\boldsymbol{A})$, occuring when V contains the minimum eigenvector. What is the <u>largest</u> possible minimum value?

If V contains $\underline{v}_1$ ($\underline{v}_1$ is eigenvector corresponding to min. eigenvalue λ_1)

$\Rightarrow R_A(\underline{v}_1) = \lambda_1 \Rightarrow \min R_A(V) = \lambda_1$



Rayleigh quotients, III

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- The minimum value of $R_{\boldsymbol{A}}(V)$ is $\lambda_{\min}(\boldsymbol{A})$, occuring when V contains the minimum eigenvector.
  What is the <u>largest</u> possible minimum value?
- The maximum value of $R_{\boldsymbol{A}}(V)$ is $\lambda_{\max}(\boldsymbol{A})$.
  What is the <u>smallest</u> possible maximum value?

Theorem (Courant-Fischer-Weyl "min-max")

Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ be Hermitian, with eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$. Then for each $1 \leqslant k \leqslant n$,

$$\lambda_k = \min_{\substack{V \subset \mathbb{C}^n \\ \dim V = k}} \max \overset{R_A}{\cancel{W}}_{\boldsymbol{A}}(V)$$

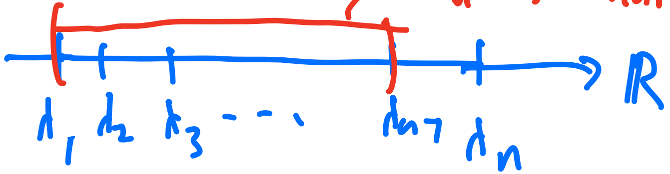
$$\lambda_k = \max_{\substack{V \subset \mathbb{C}^n \\ \dim V = n-k+1}} \min \overset{R_A}{\cancel{W}}_{\boldsymbol{A}}(V)$$

In addition, if $(\boldsymbol{u}_j)_{j=1}^n$ are the eigenvectors associated with $(\lambda_j)_{j=1}^n$, then:

- $V = \operatorname{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ *achieves the outer minimum*
- $V = \operatorname{span}\{\boldsymbol{u}_k, \ldots, \boldsymbol{u}_n\}$ *achieves the outer maximum*

$R_A(V)$ that achieves min of $\max R_A(x)$.

$k = n-1$, $\dim V = n-1$



"Proof" of 1st statement for $k=n-1$

Let V be any $(n-1)$-dim subspace

Let u_j be the eigenvector of $\underline{A}$ corresponding to λ_j.

Define $W \equiv \text{span}\{u_{n-1}, u_n\}$

$$\dim W = 2$$

$$\left.\begin{array}{l} \dim V = n-1 \\ \dim W = 2 \end{array}\right\} \dim(V \cap W) \neq 0$$

$\Rightarrow \exists\ \underline{v} \in V$ s.t. $\underline{v} = c_{n-1} u_{n-1} + c_n u_n$

c_{n-1} and c_n not both 0.

(choose $\|\underline{v}\|_2 = 1$)

$$R_A(\underline{v}) = |c_{n-1}|^2 \lambda_{n-1} + |c_n|^2 \lambda_n$$

$$\geq \lambda_{n-1}(|c_{n-1}|^2 + |c_n|^2) = \lambda_{n-1}$$

Can achieve equality if I choose V to contain u_{n-1}.

Cauchy interlacing theorem

A matrix $\boldsymbol{B}$ is a **compression** of $\boldsymbol{A}$ if $\boldsymbol{B} = \boldsymbol{Q}^* \boldsymbol{A} \boldsymbol{Q}$ for some $\boldsymbol{Q} \in \mathbb{C}^{n \times r}$ with orthonormal columns.

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Just one consequence of the min-max theorem:

Theorem (Cauchy interlacing)

Let $\boldsymbol{B} \in \mathbb{C}^{(n-1)\times(n-1)}$ be a compression of a Hermitian matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$. If $\boldsymbol{A}$ has eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$, and $\boldsymbol{B}$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$, then

$$\lambda_j \leqslant \mu_j \leqslant \lambda_{j+1},$$

for all $j = 1, \ldots, n-1$.