Math 6610: Analysis of Numerical Methods, I
Orthogonal, Projection, and Permutation matrices

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Resources: Trefethen and Bau 1997, Lectures 2, 6
Atkinson 1989, Sections 7.1
Salgado and Wise 2022, Sections 1.1, 1.2, 5.2

Unitary and orthogonal matrices

A *very* special and important class of matrices:

Definition (Unitary/orthogonal matrices)

A matrix $\boldsymbol{U} \in \mathbb{C}^{n \times n}$ is **unitary** if $\boldsymbol{U}^* \boldsymbol{U} = \boldsymbol{I}_n$

A matrix $\boldsymbol{U} \in \mathbb{R}^{n \times n}$ is **orthogonal** if $\boldsymbol{U}^T \boldsymbol{U} = \boldsymbol{I}_n$

A unitary matrix is one whose columns are an orthonormal basis for $\mathbb{C}^n$ under the $\ell^2(\mathbb{C}^n)$ inner product.

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Let $\boldsymbol{U} \in \mathbb{C}^{n \times n}$ be a unitary matrix. Then

- $\boldsymbol{U}^{-1} = \boldsymbol{U}^T$
- $\boldsymbol{U}\boldsymbol{U}^T = \boldsymbol{I}$
- $\|\boldsymbol{U}\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$ *for all* $\boldsymbol{x} \in \mathbb{C}^n$.

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This last property in particular shows: unitary matrices are *isometries*, i.e., are norm-preserving.
Unitary matrices have nice geometric interpretations: they are precisely rigid rotations and/or reflections.

Projections and projection matrices: Geometric construction

With $\mathbb{C}^n$ the ambient space, we want to define square matrix "projections". Informally, we want projections to

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Definition (Geometric definition of projection matrices)

A matrix $\boldsymbol{P} \in \mathbb{C}^{n \times n}$ is a projection matrix if

1. $\boldsymbol{P}\boldsymbol{v} = \boldsymbol{v}$ for all $\boldsymbol{v} \in V = \mathrm{range}(\boldsymbol{P})$
2. $\boldsymbol{P}\boldsymbol{w} = \boldsymbol{0}$ for all $\boldsymbol{w} \in W = \ker(\boldsymbol{P})$
3. $\mathrm{range}(\boldsymbol{P}) \oplus \ker(\boldsymbol{P}) = \mathbb{C}^n$ and $\mathrm{range}(\boldsymbol{P}) \cap \ker \boldsymbol{P} = \{0\}$.

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This last condition is *necessary*:

- A projector $\boldsymbol{P}$ in $\mathbb{R}^3$ that projects onto $(1,0,0)$ along $(0,0,1)$ is undefined for input $(0,1,0)$.
- A projector $\boldsymbol{P}$ in $\mathbb{R}^2$ that projects onto $\mathbb{R}^2$ along $(1,0)$ is ill-defined.

Projections exist and are unique

Does this definition make sense?

Theorem

If V and W are $\mathbb{C}^N$-subspaces such that $V \cap W = \{\mathbf{0}\}$ and $\dim V + \dim W = N$, then $\exists\ !$ projection matrix $\boldsymbol{P} \in \mathbb{C}^{N \times N}$ such that $\text{range}(\boldsymbol{P}) = V$ and $\ker(\boldsymbol{P}) = W$.

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Let $n = \dim V$.
If $\boldsymbol{V} \in \mathbb{C}^{N \times n}$ satisfies $\mathrm{range}(\boldsymbol{V}) = V$, and $\boldsymbol{W} \in \mathbb{C}^{N \times (N-n)}$ satisfies $\mathrm{range}(\boldsymbol{W}) = W$, then

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{V} & \mathbf{0}_{N \times (N-n)} \end{bmatrix} \begin{bmatrix} \boldsymbol{V} & \boldsymbol{W} \end{bmatrix}^{-1}$$

Projection matrices: Algebraic definitions

There is a more algebraically convenient characterization of projection matrices.

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This motivates the more common definition of a projection matrix:

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$\boldsymbol{P} \in \mathbb{C}^{n \times n}$ is a projection matrix if $\boldsymbol{P} = \boldsymbol{P}^2$.

Geometry for orthogonal projections

Projection matrices can in general inflate the size (norm) of non-trivial vectors by an arbitrary amount.

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However, projecting along $\mathrm{range}(\boldsymbol{P})^\perp$ is a norm non-expansive operation, i.e.,

$$\ker(\boldsymbol{P}) = \mathrm{range}(\boldsymbol{P})^\perp \quad \Longrightarrow \quad \|\boldsymbol{P}\boldsymbol{v}\|_2 \leqslant \|\boldsymbol{v}\|_2.$$

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Projection matrices $\boldsymbol{P}$ satisfying $\ker(\boldsymbol{P}) = (\mathrm{range}\boldsymbol{P})^\perp$ are *orthogonal* projections.

Definition (Geometric definition of orthogonal projection matrices)

Let $\boldsymbol{P}$ be a projection matrix. It's an orthogonal projector if $\boldsymbol{P} = \boldsymbol{P}^*$.

Orthogonal projectors

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Theorem

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Definition (Algebraic definition of orthogonal projectors)

$\boldsymbol{P} \in \mathbb{C}^{n \times n}$ is an orthogonal projection matrix if $\boldsymbol{P} = \boldsymbol{P}^2$ and $\boldsymbol{P} = \boldsymbol{P}^*$.

Permutation

A final class of matrices we'll consider are permutation matrices.

Definition

For a fixed $n \in \mathbb{N}$, $\pi : [n] \to [n]$ is a permutation if it is a bijection.

Permutations are linear operations, so π can be encoded as an $n \times n$ matrix.

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Permutation matrices are orthogonal/unitary, and hence if $\boldsymbol{P}$ is a permutation matrix, then $\boldsymbol{P}^* = \boldsymbol{P}^{-1}$.

The space of permutation matrices is closed under matrix multiplication, so that if $\boldsymbol{P}$ and $\boldsymbol{Q}$ are permutation matrices, then so is $\boldsymbol{P}\boldsymbol{Q}$.

References I

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