Math 6610: Analysis of Numerical Methods, I
Orthogonal, Projection, and Permutation matrices

Department of Mathematics, University of Utah

Fall 2025

Resources: Trefethen and Bau 1997, Lectures 2, 6
Atkinson 1989, Sections 7.1
Salgado and Wise 2022, Sections 1.1, 1.2, 5.2

Unitary and orthogonal matrices

A *very* special and important class of matrices:

Definition (Unitary/orthogonal matrices)

A matrix $\boldsymbol{U} \in \mathbb{C}^{n \times n}$ is **unitary** if $\boldsymbol{U}^* \boldsymbol{U} = \boldsymbol{I}_n$

A matrix $\boldsymbol{U} \in \mathbb{R}^{n \times n}$ is **orthogonal** if $\boldsymbol{U}^T \boldsymbol{U} = \boldsymbol{I}_n$

A unitary matrix is one whose columns are an orthonormal basis for $\mathbb{C}^n$ under the $\ell^2(\mathbb{C}^n)$ inner product.

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Theorem

Let $\boldsymbol{U} \in \mathbb{C}^{n \times n}$ be a unitary matrix. Then

- $\boldsymbol{U}^{-1} = \cancel{\boldsymbol{U}^T}$ U^*
- $\boldsymbol{U}\boldsymbol{U}^T = \boldsymbol{I}$ $UU^* = I$
- $\|\boldsymbol{U}\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$ *for all* $\boldsymbol{x} \in \mathbb{C}^n$.

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- $\|\boldsymbol{U}\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$ *for all* $\boldsymbol{x} \in \mathbb{C}^n$.

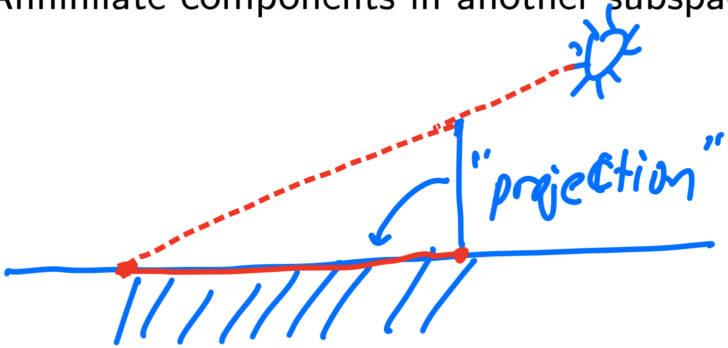
E.g.: $\underline{U} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

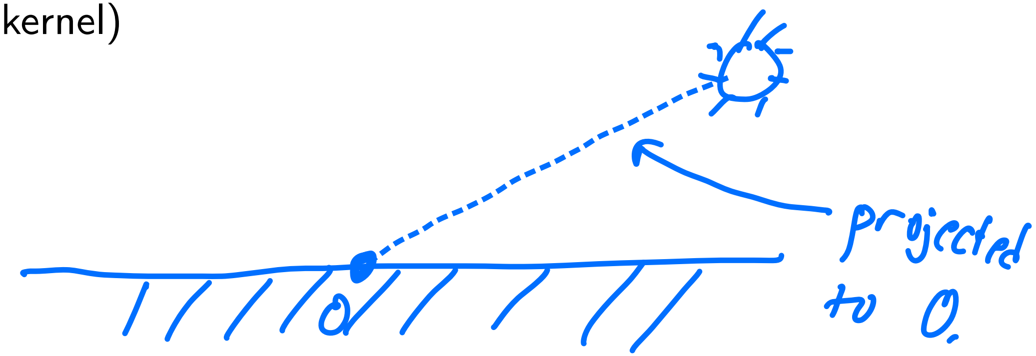
This last property in particular shows: unitary matrices are *isometries*, i.e., are norm-preserving.
Unitary matrices have nice geometric interpretations: they are precisely rigid rotations and/or reflections.

Projections and projection matrices: Geometric construction

With $\mathbb{C}^n$ the ambient space, we want to define square matrix "projections". Informally, we want projections to

- Act like the identity on some subspace (the range)
- Annihilate components in another subspace (the kernel)





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In order to be well-defined, we need an additional condition.

Definition (Geometric definition of projection matrices)

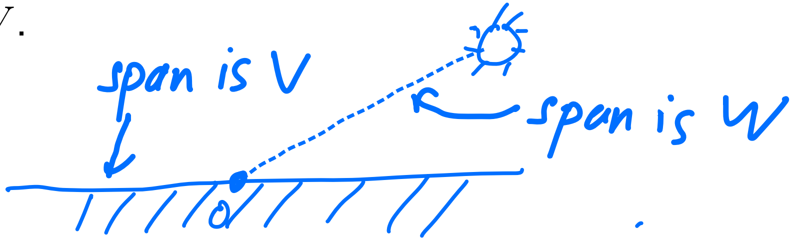
A matrix $\boldsymbol{P} \in \mathbb{C}^{n\times n}$ is a <u>projection matrix</u> if

1. $\boldsymbol{P}\boldsymbol{v} = \boldsymbol{v}$ for all $\boldsymbol{v} \in V = \mathrm{range}(\boldsymbol{P})$
2. $\boldsymbol{P}\boldsymbol{w} = \boldsymbol{0}$ for all $\boldsymbol{w} \in W = \ker(\boldsymbol{P})$
3. $\mathrm{range}(\boldsymbol{P}) \oplus \ker(\boldsymbol{P}) = \mathbb{C}^n$ and $\mathrm{range}(\boldsymbol{P}) \cap \ker \boldsymbol{P} = \{0\}$.

$V \oplus W = \{\alpha v + \beta w \mid v \in V,\ w \in W,\ \alpha, \beta \in \mathbb{C}\}$

We typically say that $\boldsymbol{P}$ projects *onto* V, and projects *along* W.

3. implies $\dim \mathrm{range}\, P + \dim \ker P = n$



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This last condition is *necessary*:

- A projector $\boldsymbol{P}$ in $\mathbb{R}^3$ that projects onto $(1,0,0)$ along $(0,0,1)$ is undefined for input $(0,1,0)$.
- A projector $\boldsymbol{P}$ in $\mathbb{R}^2$ that projects onto $\mathbb{R}^2$ along $(1,0)$ is ill-defined.

Projections exist and are unique

Does this definition make sense?

Theorem

If V and W are $\mathbb{C}^N$-subspaces such that $V \cap W = \{\mathbf{0}\}$ and $\dim V + \dim W = N$, then $\exists\,!$ projection matrix $\boldsymbol{P} \in \mathbb{C}^{N\times N}$ such that $\mathrm{range}(\boldsymbol{P}) = V$ and $\ker(\boldsymbol{P}) = W$.

Let $v_1, v_2, \ldots v_n$ be a basis for V ($\dim V = n$)

Let $w_1 \ldots w_{N-n}$ be a basis for W ($\dim W = N-n$)

$$\left.\begin{array}{ll} P v_j = v_j, & j \in [n] \\ P w_k = 0, & k \in [N-n] \end{array}\right\} \Longrightarrow P\left[v_1 \; v_2 \cdots v_n \; w_1 \cdots w_{N-n}\right] = \left[v_1 \cdots v_n \; 0 \cdots 0\right]$$

$$P(\underline{V} \; \underline{\underline{W}}) = (\underline{V} \; \underline{\underline{0}})$$

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Let $n = \dim V$.
If $\boldsymbol{V} \in \mathbb{C}^{N \times n}$ satisfies $\mathrm{range}(\boldsymbol{V}) = V$, and $\boldsymbol{W} \in \mathbb{C}^{N \times (N-n)}$ satisfies $\mathrm{range}(\boldsymbol{W}) = W$, then

$$\boldsymbol{P} = \left[\boldsymbol{V} \;\; \mathbf{0}_{N \times (N-n)}\right] \left[\boldsymbol{V} \;\; \boldsymbol{W}\right]^{-1}$$

Projection matrices: Algebraic definitions

There is a more algebraically convenient characterization of projection matrices.

A square matrix $\boldsymbol{A}$ is *idempotent* if $\boldsymbol{A} = \boldsymbol{A}^2$.

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Theorem

$\boldsymbol{P} \in \mathbb{C}^{n \times n}$ is a projection matrix iff it is idempotent.

Fact: if $V \oplus W = \mathbb{C}^n$ (and have trivial intersection)

then: $\forall x \in \mathbb{C}^n$ $x = v + w$, $v \in V$, $w \in W$, v, w are unique.

Proof: (1) Assume P is a projection matrix.

Let $x \in \mathbb{C}^n$. $\Rightarrow$ $x = v + w$, $v \in V$, $w \in W$

($V = \text{range}(P)$, $W = \ker(P)$)

$$\left.\begin{aligned} Px &= Pv + Pw = v + 0 = v \\ P^2x &= P(Px) = P(v) = v \end{aligned}\right\} P^2x = Px \quad \forall x \in \mathbb{C}^n$$

$$\Downarrow$$

$$P^2 = P$$

② Assume $P = P^2$.

Let $x \in \mathbb{C}^n$ be arbitrary: $x = \underbrace{Px}_{v} + \underbrace{(I-P)x}_{w}$

$v \in \text{range}(P)$ ✓

note: $Pw = P(I-P)x = (P - P^2)x = 0$

so $w \in \ker(P)$.

Note: $\ker(P) \cap \text{range}(P) = \{0\}$

why? If not, take $x \in \ker(P) \cap \text{range}(P)$, $x \neq 0$

$\Rightarrow Px = 0$

$x = Px + \cancel{(I-P)x}$ 0 (since $x \in \text{range}(P)$)

$x = Px$

$\parallel$

0

$Pv = v \quad \forall v \in \text{range}(P)$

$v = Pc \Rightarrow Pv = P^2c = Pc = v$

$v = Pv$ ✓

$Pw = 0 \quad \forall w \in \ker(P)$

Last thing: $\dim \text{range}(P) + \dim \ker(P) = n$

(must be true because $x = Px + (I-P)x$)

Projection matrices: Algebraic definitions

There is a more algebraically convenient characterization of projection matrices.

A square matrix $\boldsymbol{A}$ is *idempotent* if $\boldsymbol{A} = \boldsymbol{A}^2$.

Theorem

$\boldsymbol{P} \in \mathbb{C}^{n \times n}$ is a projection matrix iff it is idempotent.

This motivates the more common definition of a projection matrix:

Definition (Algebraic definition of projection matrices)

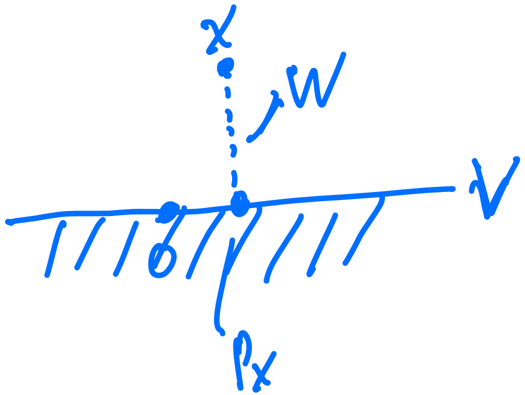
$\boldsymbol{P} \in \mathbb{C}^{n \times n}$ is a <u>projection matrix</u> if $\boldsymbol{P} = \boldsymbol{P}^2$.

Consequence: If $\underline{P}$ is a projection, range V, kernel W,
then $\underline{\underline{I}} - \underline{P}$ is a projection, range W, kernel V.

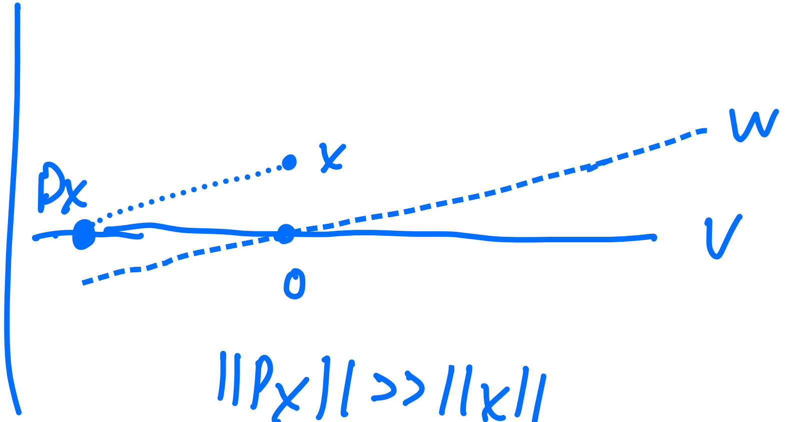
E.g.: $(I-P)^2 = I - P - P + P^2 = I - P$, $v \in V$ $(I-P)v = v - v = 0$

Geometry for orthogonal projections

Projection matrices can in general inflate the size (norm) of non-trivial vectors by an arbitrary amount.



$\|x\| \gtrsim \|Px\|$



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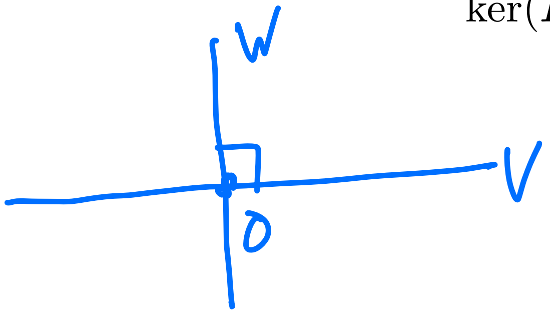
However, projecting along $\mathrm{range}(\boldsymbol{P})^{\perp}$ is a norm non-expansive operation, i.e.,

$$\ker(\boldsymbol{P}) = \mathrm{range}(\boldsymbol{P})^{\perp} \quad \Longrightarrow \quad \|\boldsymbol{P}\boldsymbol{v}\|_2 \leqslant \|\boldsymbol{v}\|_2.$$



$$\|x\|_2^2 = \|\underbrace{Px}_{\mathrm{range}(P)} + \underbrace{(I-P)x}_{\ker(P)}\|_2^2 = \|Px\|^2 + \|(I-P)x\|_2^2$$

$$\geq \|Px\|_2^2$$

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Projection matrices $\boldsymbol{P}$ satisfying $\ker(\boldsymbol{P}) = (\mathrm{range}\boldsymbol{P})^{\perp}$ are *orthogonal* projections.

Definition (Geometric definition of orthogonal projection matrices)

Let $\boldsymbol{P}$ be a projection matrix. It's an orthogonal projector ~~if $\boldsymbol{P} = \boldsymbol{P}^*$~~. range(P) ⊥ ker(P)

"orthogonal projection" ≠ "orthogonal"

Orthogonal projectors

There is also an algebraically convenient characterization of orthogonal projection matrices.

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Let $\boldsymbol{P} \in \mathbb{C}^{n \times n}$ be a projection matrix. Then it is an orthogonal projector iff it is Hermitian.

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$\boldsymbol{P} \in \mathbb{C}^{n \times n}$ is an <u>orthogonal projection matrix</u> if $\boldsymbol{P} = \boldsymbol{P}^2$ and $\boldsymbol{P} = \boldsymbol{P}^*$.

iff

Permutation

A final class of matrices we'll consider are permutation matrices.

Definition

For a fixed $n \in \mathbb{N}$, $\pi : [n] \to [n]$ is a <u>permutation</u> if it is a bijection.

Permutations are linear operations, so π can be encoded as an $n \times n$ matrix.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \xrightarrow{\pi} \begin{pmatrix} 3 \\ 1 \\ 5 \\ 4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} & & & & \\ & & ? & & \\ & & & & \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 5 \\ 4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 5 \\ 4 \\ 2 \end{pmatrix}$$

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$\boldsymbol{P} \in \mathbb{C}^{n \times n}$ is a <u>permutation matrix</u> if there is a permutation map π on $[n]$ such that $\boldsymbol{P}\boldsymbol{e}_j = \boldsymbol{e}_{\pi(j)}$ for all $j \in [n]$.

$$\underline{e}_j = \begin{array}{c} \\ \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array} \right) \end{array} \begin{array}{l} \\ \\ \\ \leftarrow j \\ \\ \\ \\ \end{array}$$

Columns of P are orthonormal

$\Rightarrow P$ is unitary

$\Rightarrow P^{-1} = P^*$

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Permutation matrices are orthogonal/unitary, and hence if $\boldsymbol{P}$ is a permutation matrix, then $\boldsymbol{P}^* = \boldsymbol{P}^{-1}$.

The space of permutation matrices is closed under matrix multiplication, so that if $\boldsymbol{P}$ and $\boldsymbol{Q}$ are permutation matrices, then so is $\boldsymbol{P}\boldsymbol{Q}$.

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