Department of Mathematics, University of Utah

Analysis of Numerical Methods I

MATH 6610 – Section 001 – Fall 2025

Homework 3

Eigenvalues

Due Wednesday, September 10, 2025

Submission instructions:
Submit your assignment on gradescope.

Problem assignment:

1. Using properties of the trace and determinant (notably that $\det(\boldsymbol{AB}) = (\det \boldsymbol{A})(\det \boldsymbol{B})$ and $\operatorname{tr}(\boldsymbol{AB}) = \operatorname{tr}(\boldsymbol{BA})$), show that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\boldsymbol{A} \in \mathbb{C}^{n \times n}$, then,

$$\det \boldsymbol{A} = \prod_{i \in [n]} \lambda_i, \qquad \operatorname{tr} \boldsymbol{A} = \sum_{i \in [n]} \lambda_i.$$

2. (Gershgorin's theorem) Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$, with entries $(A)_{i,j} = a_{i,j}$. Define the following *Gershgorin discs* in the complex plane:

$$D_i \coloneqq \left\{ z \in \mathbb{C} \;\middle|\; |z - a_{i,i}| \le \sum_{j \neq i} |a_{i,j}| \right\}, \qquad i \in [n].$$

Prove that $\lambda(\boldsymbol{A}) \subset \bigcup_{i \in [n]} D_i$. (Pick an eigenpair $(\lambda, \boldsymbol{w})$, and algebraically manipulate a particular row of the equality $\boldsymbol{A}\boldsymbol{w} = \lambda \boldsymbol{w}$.)

3. (The spectral theorem for skew-Hermitian matrices) Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ be skew-Hermitian, i.e., $\boldsymbol{A}^* = -\boldsymbol{A}$. Prove that $\boldsymbol{A}$ is skew-Hermitian iff it's unitarily diagonalizable with purely imaginary spectrum.

4. (Schur decomposition) Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$. Prove that there exists a unitary matrix $\boldsymbol{U} \in \mathbb{C}^{n \times n}$ and an upper-triangular matrix $\boldsymbol{T}$ such that $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{T}\boldsymbol{U}^*$. (Start with $\boldsymbol{U}_1$ any unitary matrix where the first column is a unit-norm eigenvector of $\boldsymbol{A}$, and compute $\boldsymbol{U}_1^* \boldsymbol{A} \boldsymbol{U}_1$.)

5. (Numerical range) Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$. Define $r(\boldsymbol{A}) = \sup |W(\boldsymbol{A})|$, which is the *numerical radius* of $\boldsymbol{A}$.

(a) Show that $\rho(\boldsymbol{A}) \le r(\boldsymbol{A}) \le \|\boldsymbol{A}\|_2$, where $\rho(\boldsymbol{A})$ is the spectral radius of $\boldsymbol{A}$.
(b) Let $\theta \in [0, 2\pi)$. Show that $W(e^{i\theta}\boldsymbol{A}) = e^{i\theta} W(\boldsymbol{A})$.
(c) Show that $W(\boldsymbol{A}^*) = (W(\boldsymbol{A}))^*$.
(d) Define $H(\boldsymbol{A}) \coloneqq \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^*)$ as the *Hermitian part* of $\boldsymbol{A}$. Describe how $W(H(\boldsymbol{A}))$ relates to $W(\boldsymbol{A})$. How would you compute $W(H(\boldsymbol{A}))$?

6. (Computing the numerical range of a matrix) Using the facts about $W(\boldsymbol{A})$ in the previous problem, devise a feasible algorithm for computing the numerical range of a general matrix $\boldsymbol{A} \in \mathbb{C}^{n\times n}$. You may (and should) rely on the ability to compute the spectrum of Hermitian matrices, which is a routine provided in every modern numerical computing software, language, or library. Note that since $W(\boldsymbol{A})$ is compact, you need only compute points on the boundary of the set. Explain your algorithm at the level of a discussion of pseudocode implementation.

Implement your algorithm (in your software of choice) by testing on various matrices $\boldsymbol{A}$. In particular, computationally investigate the numerical range of the $n \times n$ nilpotent matrix that is all zero except for entries 1 on the main superdiagonal. (This is a size-n Jordan block.) If $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ is an arbitrary matrix, how would you describe $W(\boldsymbol{J})$, the numerical range of its Jordan normal form?