Department of Mathematics, University of Utah

Analysis of Numerical Methods I
MATH 6610 – Section 001 – Fall 2025
Homework 2
Unitary and projection matrices

Due Wednesday, September 3, 2025

Submission instructions:
Submit your assignment on gradescope.

Problem assignment:

1. (Triangular, unitary matrices) Show that if a matrix is both triangular and unitary, then it must be diagonal.

2. (Unitarily invariant norms) A norm $\|\cdot\|$ on matrices $\boldsymbol{A}$ is *unitarily invariant* if $\|\boldsymbol{UA}\| = \|\boldsymbol{AV}\| = \|\boldsymbol{A}\|$ for all unitary matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ of conforming sizes.

(a) If $\boldsymbol{U} \in \mathbb{C}^{n\times n}$ is unitary, prove that $\|\boldsymbol{Ux}\|_2 = \|\boldsymbol{x}\|_2$ for any $\boldsymbol{x} \in \mathbb{C}^n$.

(b) Prove that the induced 2-norm on matrices of arbitrary size is unitarily invariant.

(c) Prove that the Frobenius norm on matrices of arbitrary size is unitarily invariant.

(d) Let $\boldsymbol{W} \in \mathbb{C}^{m\times n}$ satisfy $\boldsymbol{W}^*\boldsymbol{W} = \boldsymbol{I}$. Prove that $\|\boldsymbol{Wx}\|_2 = \|\boldsymbol{x}\|_2$ for every $\boldsymbol{x} \in \mathbb{C}^n$. (Note here that $\boldsymbol{x}$ and $\boldsymbol{Wx}$ are vectors of potentially different sizes.)

3. (Algebraic definition of orthogonal projectors) From the geometric definition of orthogonal projections (see slide `D02-S06(c)`), prove that a projection matrix is an orthogonal projector iff it's Hermitian. (This question is asking you to prove the theorem on slide `D02-S07`.)

4. (Uniqueness of projections) Let $\boldsymbol{P}$ be a projection matrix corresponding to range V and kernel W. Prove that $\boldsymbol{P}$ is unique, i.e., that any other projection matrix with the same range and kernel must be $\boldsymbol{P}$.

5. (Sherman-Morrison formula) Let $\boldsymbol{u}$ and $\boldsymbol{w}$ be nonzero vectors in $\mathbb{C}^n$.

(a) Show that $\boldsymbol{I} + \boldsymbol{uw}^*$ is singular iff $\boldsymbol{w}^*\boldsymbol{u} = -1$.

(b) Show that, if $\boldsymbol{w}^*\boldsymbol{u} \neq -1$, then $(\boldsymbol{I} + \boldsymbol{uw}^*)^{-1} - \boldsymbol{I}$ is a rank-1 matrix.

(c) If $\boldsymbol{w}^*\boldsymbol{u} \neq -1$, give an explicit formula for the inverse of the rank-1 pertubation $\boldsymbol{I} + \boldsymbol{uw}^*$ of $\boldsymbol{I}$, as a rank-1 perturbation of $\boldsymbol{I}^{-1} = \boldsymbol{I}$.

(d) Let $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ be invertible. Give a sufficient and necessary condition for $\boldsymbol{A} + \boldsymbol{uw}^*$ to be invertible, and when this is invertible, provide a formula explicitly showing that its inverse is a rank-1 perturbation of $\boldsymbol{A}^{-1}$.

6. (Norms of projections) Let $M \in [1, \infty)$ and $n \geq 2$ be arbitrary. Explicitly construct a projection matrix $\boldsymbol{P} \in \mathbb{C}^{n\times n}$ such that $\|\boldsymbol{P}\|_2 = M$.

7. Let $\boldsymbol{P} \in \mathbb{C}^{n \times n}$ be a projection matrix. Identify, with justification, the spectrum of $\boldsymbol{P}$ (including the algebraic multiplicity of the eigenvalues). Give sufficient and necessary conditions for $\boldsymbol{P}$ to be diagonalizable, and when these conditions hold, derive an eigendecomposition for $\boldsymbol{P}$ in terms of the range and kernel of $\boldsymbol{P}$.

8. (Spectrum of Kronecker products) Let $\boldsymbol{A} \in \mathbb{C}^{m \times m}$ and $\boldsymbol{B} \in \mathbb{C}^{n \times n}$ be square matrices. The *Kronecker product* $\boldsymbol{A} \otimes \boldsymbol{B}$ is an $mn \times mn$ matrix that is a tiling of $\boldsymbol{B}$ scaled by the entries of $\boldsymbol{A}$. With $a_{j,k} = (A)_{j,k}$,

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{pmatrix} a_{1,1}\boldsymbol{B} & a_{1,2}\boldsymbol{B} & \cdots & a_{1,m}\boldsymbol{B} \\ a_{2,1}\boldsymbol{B} & a_{2,2}\boldsymbol{B} & \cdots & a_{2,m}\boldsymbol{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}\boldsymbol{B} & a_{m,2}\boldsymbol{B} & \cdots & a_{m,m}\boldsymbol{B} \end{pmatrix} \in \mathbb{C}^{mn \times mn}$$

(Even for rectangular matrices, $\boldsymbol{A} \otimes \boldsymbol{B}$ is well-defined as a tiling of $\boldsymbol{B}$ scaled by the entries of $\boldsymbol{A}$.) Assume $\boldsymbol{A}$ and $\boldsymbol{B}$ are both diagonalizable with given spectrum and eigenvectors. Compute both the spectrum and the eigenvectors of $\boldsymbol{A} \otimes \boldsymbol{B}$ in terms of those of $\boldsymbol{A}$ and $\boldsymbol{B}$.