Department of Mathematics, University of Utah
**Analysis of Numerical Methods I**
**MATH 6610 – Section 001 – Fall 2025**
**Homework 12**
**Fourier series approximation**

**Due Wednesday, November 26, 2025**

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**Submission instructions**:
Submit your assignment on gradescope.

**Problem assignment**:

**1.** Prove that if $u \in H_p^s$ for some integer $s \geq 0$, then

$$\|u - P_N u\|_{L^2} \leq N^{-s} \|u\|_{H_p^s},$$

where the space $H_p^s$ is defined on slide `D12-S12` and $P_N$ is defined on slide `D12-S08(b)`.

**2.** (The Lebesgue constant) Define $C([0, 2\pi]; \mathbb{C})$ as the space of complex-valued continuous functions on $[0, 2\pi]$. This space is metrized with the norm,

$$f \in C = C([0, 2\pi]; \mathbb{C}) \quad \Longrightarrow \quad \|f\|_C = \max_{x \in [0,2\pi]} |f(x)|.$$

Let $N \in \mathbb{N}_0$, and define $M = 2N + 1$. Now let $X_M \coloneqq \{x_1, \ldots, x_M\} \subset [0, 2\pi]$ be $M$ distinct points on $[0, 2\pi]$, and consider the interpolation operator $I_M : C \to V_N$, where for any $f \in C$:

$$V_N = \operatorname{span}\{e^{ikx} \;\big|\; k \in \mathbb{Z},\ |k| \leq N\}, \qquad\qquad (I_M f)(x_m) = f(x_m).$$

Assume that $I_N$ is unisolvent on $X_M$. Prove that $\Lambda$, defined as,

$$\Lambda \coloneqq \|I_M\|_{C \mapsto C} = \sup_{f \in C,\, f \neq 0} \frac{\|I_M f\|_C}{\|f\|_C}$$

is given by,

$$\Lambda = \max_{x \in [0,2\pi]} \lambda(x), \qquad\qquad \lambda(x) = \sum_{m \in [M]} |\ell_m(x)|,$$

where $\{\ell_m\}_{m \in [M]}$ is the set of cardinal Lagrange functions in $V_N$ associated to the interpolation points $X_M$. $\Lambda$ is called the *Lebesgue constant* (associated to $(V_N, X_M)$), and $\lambda$ is the *Lebesgue function*.

**3.** (Lebesgue's Lemma) Let $V$ be an $M$-dimensional subspace of functions on $\mathbb{R}$, and let $X$ be a set of $M$ distinct points on $\mathbb{R}$. Assume the interpolation problem on the nodes $X$ and the subspace $V$ is unisolvent, and hence that a corresponding $(V, X)$ interpolation operator $I_M$ is well-defined. Prove that for any continuous function $f$,

$$\|f - I_M f\| \leq (1 + \Lambda) \inf_{v \in V} \|f - v\|,$$

where $\Lambda$ is the $(V, X)$ Lebesgue constant, and $\|\cdot\|$ is the maximum norm on the domain, as in the previous problem.

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**4.** (The DFT Lebesgue constant) Fix $N \in \mathbb{N}_0$ with $x \in [0, 2\pi]$, and with $M = 2N + 1$ consider the $M$-point interpolant onto the space $V_N = \text{span}\{e^{ikx}\}_{|k| \leq N}$. Let $X_M = \{x_m\}_{m \in [M]}$ be the equidistant nodes $x_m = 2\pi(m-1)/M$. Consider the interpolation operator $I_M$ associated with $(V_N, X_M)$.

(a) Show that the cardinal Lagrange function centered at the point $x_j$ is given by,

$$\ell_j(x) = \frac{1}{M} \frac{\sin\left(\frac{M}{2}(x - x_j)\right)}{\sin\left(\frac{1}{2}(x - x_j)\right)}.$$

(This function is called the normalized *Dirichlet kernel.*)

(b) For this choice of nodal set, show that the Lebesgue constant $\Lambda$, defined earlier, satisfies $\Lambda \gtrsim \log M$, and hence that $\Lambda$ cannot be bounded as $M \uparrow \infty$.
(One way to do this is to evaluate the Lebesgue function at a midpoint between two consecutive gridpoints. Here $a \gtrsim b$ means $a \geq Cb$ for some universal constant $C$. The asymptotic lower bound above is actually the correct asymptotic behavior.)

**5.** (Fourier interpolation)

(a) For any $N \geq 1, k \in \mathbb{Z}$, prove that $I_N \phi_k = \phi_\ell$, where $\ell$ satisfying $|\ell| \leq N$ is the modular restriction of $k$ to $[-N, N]$:

$$\ell = \ell(k) \coloneqq k - (2N+1)j \in [-N, N], \qquad\qquad j \in \mathbb{Z}.$$

An equivalent definition: $\ell(k) = -N + [(k + N) \pmod{2N+1}]$. The operator $I_N$ is defined on `D13-S12(a)` as the $M = (2N+1)$-point Fourier interpolation operator, and $\phi_k$ is defined on slide `D13-S02(a)`.

(b) If $u \in H_p^s$, prove that,

$$\|u - I_N u\|_{L^2} \lesssim N^{-s} \|u\|_{H_p^s},$$

where $a \lesssim b$ means that $a \leq Cb$ for some constant $C$ independent of $N$ and $u$.

**6.** Numerically confirm the behavior stated by the theorems on slides `D12-S13(a)` ($L^2$ convergence of Fourier projection) and `D13-S16(a)` ($L^2$ convergence of Fourier interpolation; don't worry about $H^r$ convergence). To do this, for the first theorem, numerically compute $\|u_j - P_N u_j\|_{L^2}$ as a function of $N$ for each $j = 0, 1, 2, 3$. The functions $u_j$, $j \geq 0$, are defined as,

$$u_0(x) \coloneqq \begin{cases} 1, & |x - \pi| < \frac{\pi}{2} \\ -1, & \textit{else} \end{cases} \qquad\qquad u_j(x) \coloneqq c_j + \int_0^x u_{j-1}(y)\, \mathrm{d}y \quad (j \geq 1),$$

where $c_j$ is chosen so that $u_j$ is a mean-0 function. Based on your numerical results what type of regularity ($s$) does $u_j$ seem to have? Repeat the experiment for the error in $I_N u_j$.