Department of Mathematics, University of Utah

Analysis of Numerical Methods I

MATH 6610 – Section 001 – Fall 2025

Homework 11

Nonlinear equations

Due Wednesday, November 19, 2025

Submission instructions:
Submit your assignment on gradescope.

Problem assignment:

1. (Fixed-point iteration) Let $\boldsymbol{f} : D \to D$ be Lipschitz continuous over some domain $D \subset \mathbb{R}^n$ with Lipschitz constant $L < 1$.

(a) Show that there cannot be two distinct fixed points. I.e., there cannot be points $\boldsymbol{x}, \boldsymbol{y} \in D$, $\boldsymbol{x} \neq \boldsymbol{y}$ such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$ and $\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{y}$.

(b) Prove that the fixed point iteration $\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k)$ converges to a fixed point of $\boldsymbol{f}$ for any starting point $\boldsymbol{x}_0 \in D$.

(You may use the fact that Cauchy sequences converge to a limit in $\|\cdot\|_2$ on $\mathbb{R}^n$. A sequence of vectors $\{\boldsymbol{x}_k\}_{k\geq 0}$ is Cauchy if, given any $\epsilon > 0$, there is some $K = K(\epsilon)$ such that for any $n, m \geq K$, then $\|\boldsymbol{x}_m - \boldsymbol{x}_n\|_2 < \epsilon$.)

2. (Convergence of Newton's method) Let $f : \mathbb{R} \to \mathbb{R}$ have a root at x_*. Assume that $f'(x_*) \neq 0$, and that f'' is continuous. Prove that Newton's method for rootfinding converges to the root x_* so long as the initial guess x_0 is in some neighborhood of x_*.
(You need not show that convergence is quadratic, only that it does converge.)

3. (Newton's method for optimization) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Consider solving the optimization problem $\min_{\boldsymbol{x}} f(\boldsymbol{x})$. A necessary condition for $\boldsymbol{x}$ to correspond to an optimal value of f is that $\nabla f(\boldsymbol{x}) = \frac{\mathrm{d}f}{\mathrm{d}\boldsymbol{x}} = \boldsymbol{0}$. Write down Newton's method for numerically computing a solution to this system of equations.

4. Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$.

(a) If $m = n$ and $\boldsymbol{A}$ is invertible and $\boldsymbol{b} \in \mathbb{R}^n$ is given, write down Newton's method for finding the solution of the system of equations $\boldsymbol{f}(\boldsymbol{x}) \coloneqq \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} = \boldsymbol{0}$. Does it converge to the exact solution? Estimate how many iterations are required before an acceptable approximation is achieved.

(b) If $m \geq n$ and $\boldsymbol{A}$ has full column rank, let $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}$, and $f(\boldsymbol{x}) = \|\boldsymbol{g}(\boldsymbol{x})\|_2^2$. Write down Newton's method for optimization corresponding to the necessary optimality conditions in the previous problem. Does it converge to the exact solution? Estimate how many iterations are required before an acceptable approximation is achieved.

5. (Jacobi method and relaxation) Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be the explicit nearly-tridiagonal matrix,

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix}$$

Consider the Jacobi (stationary iterative) method applied to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

(a) Does this iterative procedure converge? What assumptions/conditions are needed, especially on $\boldsymbol{x}_0$ and/or $\boldsymbol{b}$?

(b) Consider solving $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$, where clearly one solution is $\boldsymbol{x} = 0$, and assume n is odd. Fix a frequency $k \in \mathbb{Z}$, $|k| < n/2$, and let $\boldsymbol{x}_0$ be the vector with entries $(x_0)_j = e^{ij2\pi k/n}$, where i is the imaginary unit. Show that $\|\boldsymbol{x}_m\|_2 \leq \rho^m \|\boldsymbol{x}_0\|_2$, where $\rho = \rho(k)$, and determine a formula for ρ.

(c) Informally, how does this procedure perform when the frequency magnitude $|k|$ is large (say close to $n/2$) versus when it's small (say close to 0)? If $\boldsymbol{x}_0$ is given as a sum of frequencies, what behavior do you expect?