Department of Mathematics, University of Utah

Analysis of Numerical Methods I
MATH 6610 – Section 001 – Fall 2025
Homework 10
Iterative methods for linear systems

Due Wednesday, November 12, 2025

Submission instructions:
Submit your assignment on gradescope.

Problem assignment:

1. (Contractivity of iterated matrices)

(a) Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ and let $\|\cdot\|$ be any norm on matrices. Show that, even if $\rho(\boldsymbol{A}) < 1$, it's possible that $\lim_{k\to\infty} \|\boldsymbol{A}\|^k \neq 0$. (Here $\rho(\boldsymbol{A})$ is the spectral radius of $\boldsymbol{A}$.)

(b) Let $\boldsymbol{N} \in \mathbb{C}^{q \times q}$ be a size-q Jordan block, ($q \in \mathbb{N}$) associated with the eigenvalue 0. I.e., $(N)_{j,j+1} = 1$ for $j \in [q-1]$, and all other entries are 0. Prove that $\boldsymbol{N}^q = \boldsymbol{0}$. ($\boldsymbol{N}$ is called a *nilpotent matrix*, and q is the *degree of nilpotency*.)

(c) Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$. Prove that $\rho(\boldsymbol{A}) < 1$ iff $\lim_{k\to\infty} \boldsymbol{A}^k = \boldsymbol{0}$.

2. (Neumann Series) Suppose that $\boldsymbol{A} \in \mathbb{C}^{n \times n}$. Prove that $\rho(\boldsymbol{A}) < 1$ iff

$$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \sum_{k=0}^{\infty} \boldsymbol{A}^k.$$

(You may find it helpful to recall how the geometric series is proven when $\boldsymbol{A}$ is a scalar.)

3. (Linear fixed point iteration) Let $\boldsymbol{A} = \boldsymbol{B} + \boldsymbol{C} \in \mathbb{C}^{n \times n}$, with $\boldsymbol{A}$ and $\boldsymbol{B}$ invertible. Prove that $\rho(\boldsymbol{B}^{-1}\boldsymbol{C}) < 1$ iff the iterates $\boldsymbol{x}_k$ of the fixed point iteration,

$$\boldsymbol{B}\boldsymbol{x}_{k+1} = \boldsymbol{b} - \boldsymbol{C}\boldsymbol{x}_k, \qquad k \geq 0,$$

converge to $\boldsymbol{A}^{-1}\boldsymbol{b}$ for fixed but arbitrary $\boldsymbol{b}, \boldsymbol{x}_0 \in \mathbb{C}^n$.

4. (Jacobi iteration for diagonally dominant systems) A matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ is (strictly) *diagonally dominant* if

$$|A_{j,j}| > \sum_{k \neq j} |A_{j,k}|, \qquad j \in [n].$$

Asssume $\boldsymbol{A}$ is diagonally dominant, and let $\boldsymbol{b} \in \mathbb{C}^n$ be given. Show that the Jacobi stationary iterative method (cf. slide `D10-S10(b)`) applied to the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ converges to the exact solution $\boldsymbol{x}$ for any initial guess $\boldsymbol{x}_0$.

5. (Stationary iterative methods for singular matrices) Let $\boldsymbol{G} \in \mathbb{C}^{n \times n}$ satisfy $\rho(\boldsymbol{G}) = 1$, with a non-defective eigenvalue $\lambda = 1$, and all other eigenvalues η satisfy $|\eta| < 1$.

(a) For arbitrary $\boldsymbol{x} \in \mathbb{C}^n$, prove that $\lim_{k\to\infty} \boldsymbol{G}^k \boldsymbol{x}$ converges, and determine what vector it converges to.

(b) Precisely prescribe the set of valid $\boldsymbol{y} \in \mathbb{C}^n$ such that $\sum_{j=0}^{\infty} \boldsymbol{G}^j \boldsymbol{y}$ converges. (Again, explicitly identify what it converges to.)

(c) Assume $\boldsymbol{B}, \boldsymbol{C} \in \mathbb{C}^{n\times n}$, with $\boldsymbol{B}$ invertible, are such that $\boldsymbol{G} := -\boldsymbol{B}^{-1}\boldsymbol{C}$ satisfies the assumptions in this problem. Show that, with this setup, the fixed point iterative method in problem 3 corresponds to a singular matrix $\boldsymbol{A} = \boldsymbol{B} + \boldsymbol{C}$. Prove that, nevertheless, and under particular assumptions on $\boldsymbol{x}_0$ and/or $\boldsymbol{b}$, this iterative scheme converges, and identify what it converges to.