Department of Mathematics, University of Utah
Applied Complex Variables and Asymptotic Methods
MATH 6720 – Section 001 – Spring 2024
Homework 7
Asymptotic Methods

Due: Friday, April 19, 2024

Below, problem C in section A.B is referred to as exercise A.B.C.
Text: *Complex Variables: Introduction and Applications*, Ablowitz & Fokas,

Exercises: 5.7.1
5.7.5
6.1.1
6.1.2
6.2.2
6.2.3
6.2.5
6.3.1
6.3.3

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5.7.1. Show that the "cross ratios" associated with the points $(z, 0, 1, -1)$ and $(w, i, 2, 4)$ are $(z+1)/2z$ and $(w-4)(2-i)/2(i-w)$, respectively. Use these to find the bilinear transformation that maps $0, 1, -1$ to $i, 2, 4$.

5.7.5. Let C_1 be the circle with center $i/2$ passing through 0, and let C_2 be the circle with center $i/4$ passing through 0 (see Figure 5.7.7 in the text). Let D be the region enclosed by C_1 and C_2. Show that the inversion $w_1 = 1/z$ maps D onto the strip $-2 < \mathrm{Im}\,(w)_1 < -1$ and the transformation $w_2 = e^{\pi w_1}$ maps this strip to the upper half plane. Use these results to find a conformal mapping that maps D onto the unit disk.

6.1.1.

(a) Consider the function

$$f(\epsilon) = e^{\epsilon}.$$

Find the asymptotic expansion of $f(\epsilon)$ in powers of ϵ as $\epsilon \to 0$.

(b) Similarly for the function

$$f(\epsilon) = e^{-\frac{1}{\epsilon}},$$

find the asymptotic expansion of $f(\epsilon)$ in powers of ϵ as $\epsilon \to 0$.

6.1.2. Show that both the functions $(1+x)^{-1}$ and $(1+e^{-x})(1+x)^{-1}$ possess the same asymptotic expansion as $x \to \infty$.

6.2.2. Use integration by parts to obtain the first two terms of the asymptotic expansion of

$$\int_1^\infty e^{-k(t^2+1)}\,\mathrm{d}t.$$

6.2.3. Use Watson's Lemma to obtain an infinite asymptotic expansion of

$$I(k) = \int_0^\pi e^{-kt} t^{-\frac{1}{3}} \cos t \,\mathrm{d}t,$$

as $k \to \infty$.

6.2.5. Use Laplace's method to determine the leading behavior (first term) of

$$I(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-k \sin^4 t}\,\mathrm{d}t$$

as $k \to \infty$.

6.3.1. Use integration by parts to obtain the asymptotic expansion as $k \to \infty$ of the following integrals up to order $\frac{1}{k^2}$:
(a) $\int_0^2 (\sin t + t) e^{ikt}\,\mathrm{d}t$
(b) $\int_0^\infty \frac{e^{ikt}}{1+t^2}\,\mathrm{d}t$

6.3.3. Use the method of stationary phase to find the leading behavior of the following integrals as $k \to \infty$:
(a) $\int_0^1 \tan t \; e^{ikt^4}\,\mathrm{d}t$
(b) $\int_{\frac{1}{2}}^2 (1+t) e^{ik\left(\frac{t^3}{3} - t\right)}\,\mathrm{d}t$