Department of Mathematics, University of Utah

Applied Complex Variables and Asymptotic Methods

MATH 6720 – Section 001 – Spring 2024

Homework 6

Residue Calculus, II

Due: Friday, March 29, 2024

Below, problem C in section A.B is referred to as exercise A.B.C.

Text: *Complex Variables: Introduction and Applications*, Ablowitz & Fokas,

Exercises:
- 4.2.1, parts (b) and (d)
- 4.2.2, parts (a), (d), and (g)
- 4.2.5
- 4.2.7
- 4.3.2
- 4.3.3
- 4.3.7, part (a) only. Note that $0 < k < 1$ is the correct restriction on k.
- 4.3.13, only compute the first integral, i.e., the one involving $x^{1/2} \log x$. In addition, for this section the text considers the principal branch of $\log z$ and $z^{1/2}$ to correspond to $z = re^{i\theta}$ for $\theta \in [0, 2\pi)$.

Submit your homework assignment on Canvas via Gradescope.

4.2.1. Evaluate the following real integrals.

(b) $\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}$, $a^2 > 0$

(d) $\int_0^\infty \frac{\mathrm{d}x}{x^6+1}$

4.2.2. Evaluate the following real integrals by residue integration:

(a) $\int_{-\infty}^\infty \frac{x \sin x}{x^2+a^2}\,\mathrm{d}x$, $\quad a^2 > 0$

(d) $\int_0^\infty \frac{\cos kx}{x^4+1}\,\mathrm{d}x$, $\quad k$ real

(g) $\int_0^{\pi/2} \sin^4\theta\,\mathrm{d}\theta$

4.2.5. Consider a rectangular contour with corners at $b \pm iR$ and $b + 1 \pm iR$. Use this contour to show that,

$$\lim_{R\to\infty} \frac{1}{2\pi i}\int_{b-iR}^{b+iR} \frac{e^{az}}{\sin \pi z}\,\mathrm{d}z = \frac{1}{\pi(1+e^{-a})},$$

where $0 < b < 1$ and $|\mathrm{Im}\,(a)\,| < \pi$.

4.2.7. Use a sector contour with radius R, as in Figure 4.2.6 in the text, centered at the origin with angle $0 \leq \theta \leq \frac{2\pi}{5}$ to find, for $a > 0$,

$$\int_0^\infty \frac{\mathrm{d}x}{x^5 + a^5} = \frac{\pi}{5a^4 \sin \frac{\pi}{5}}.$$

4.3.2. Show that,

$$\int_0^\infty \frac{\sin x}{x(x^2+1)} \, \mathrm{d}x = \frac{\pi}{2} \left(1 - \frac{1}{e}\right).$$

4.3.3. Show that,

$$\int_{-\infty}^\infty \frac{\cos x - 1}{x^2(x^2+a^2)} \, \mathrm{d}x = -\frac{\pi}{a^2} + \frac{\pi}{a^3} \left(1 - e^{-a}\right), \quad a > 0.$$

4.3.7. Use the keyhole contour of Figure 4.3.6 in the text to show that on the principal branch of x^k,

(a)

$$I(a) = \int_0^\infty \frac{x^{k-1}}{(x+a)} \, \mathrm{d}x = \frac{\pi}{\sin k\pi} a^{k-1}, \quad 0 < k < 1, \quad a > 0$$

4.3.13. Use the keyhole contour of Figure 4.3.6 to show for the principal branch of $x^{1/2}$ and $\log x$,

$$\int_0^\infty \frac{x^{1/2} \log x}{(1+x^2)} \, \mathrm{d}x = \frac{\pi^2}{2\sqrt{2}}$$