Department of Mathematics, University of Utah

Applied Complex Variables and Asymptotic Methods
MATH 6720 – Section 001 – Spring 2024
Homework 6 Solutions
Residue Calculus, II

Due: Friday, March 29, 2024

Below, problem C in section A.B is referred to as exercise A.B.C.

Text: *Complex Variables: Introduction and Applications*, Ablowitz & Fokas,

Exercises:
- 4.2.1, parts (b) and (d)
- 4.2.2, parts (a), (d), and (g)
- 4.2.5
- 4.2.7
- 4.3.2
- 4.3.3
- 4.3.7, part (a) only. Note that $0 < k < 1$ is the correct restriction on k.
- 4.3.13, only compute the first integral, i.e., the one involving $x^{1/2}\log x$. In addition, for this section the text considers the principal branch of $\log z$ and $z^{1/2}$ to correspond to $z = re^{i\theta}$ for $\theta \in [0, 2\pi)$.

Submit your homework assignment on Canvas via Gradescope.

4.2.1. Evaluate the following real integrals.
(b) $\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}$, $a^2 > 0$
(d) $\int_0^\infty \frac{\mathrm{d}x}{x^6+1}$

Solution: The main technique will be using the Cauchy Residue Theorem on a closed contour that is the union of the real interval $[-R, R]$ with a circular contour C_R, with C_R defined as the portion of $\partial B_R(0)$ in the upper half-plane. I.e., for a suitably defined $f(z)$ with singularities $\{z_j\}_{j=1}^M$ in the upper half-plane, we will compute via the Cauchy Residue Theorem,

$$\lim_{R\uparrow\infty}\left[\int_{-R}^{R} f(z)\,\mathrm{d}z + \int_{C_R} f(z)\,\mathrm{d}z\right] = 2\pi i \sum_{j=1}^{M} \mathrm{Res}(f; z_j).$$

In this case, we will have,

$$\lim_{R\uparrow\infty}\int_{C_R} f(z)\,\mathrm{d}z = 0 \quad \Longrightarrow \quad \mathrm{PV}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_{j=1}^{M} \mathrm{Res}(f; z_j), \tag{1}$$

and the equality above will be our main strategy for computing these integrals.

(b) We assume without loss that $a > 0$ (since $a \leftarrow -a$ leaves the integral unchanged). Since the integrand is even, then

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2} = \frac{1}{2}\mathrm{PV}\int_{-\infty}^\infty f(x)\,\mathrm{d}x, \quad \text{where } f(x) \coloneqq \frac{1}{x^2+a^2}$$

(The principal value is not needed here, but we'll continue to use it.) With C_R the circular contour described above, we have

$$\lim_{R\uparrow\infty}\int_{C_R} f(z)\,\mathrm{d}x = 0,$$

since f is a rational function of z with $f(z) = P(z)/Q(z)$ and $\deg Q \geq \deg P + 2$. There is a lone singularity of f in the upper half-plane at $z = ia$, which is a pole of order 2 with residue,

$$2\pi i \mathrm{Res}(f; ia) = 2\pi i \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}z}\left((z-ia)^2 f(z)\right)\Big|_{z=ia} = 2\pi i \frac{-2}{(ia+ia)^3} = \frac{\pi}{2a^3}.$$

Using these in (1) yields,

$$\mathrm{PV}\int_{-\infty}^\infty f(z)\,\mathrm{d}z = \frac{\pi}{2a^3},$$

and therefore,

$$\int_0^\infty \frac{1}{(x^2+a^2)^2}\,\mathrm{d}x = \frac{\pi}{4a^3}, \quad a > 0,$$

and thus for arbitrary real $a \neq 0$, we have

$$\int_0^\infty \frac{1}{(x^2+a^2)^2}\,\mathrm{d}x = \frac{\pi}{4|a|^3}.$$

(d) Since the integrand is even, then

$$\int_0^\infty \frac{\mathrm{d}x}{x^6+1} = \frac{1}{2}\mathrm{PV}\int_{-\infty}^\infty f(x)\,\mathrm{d}x, \quad \text{where } f(x) \coloneqq \frac{1}{x^6+1}$$

(Again, the principal value is not really needed here.) With C_R the circular contour described above, we have

$$\lim_{R\uparrow\infty}\int_{C_R} f(z)\,\mathrm{d}x = 0,$$

since f is a rational function of z with $f(z) = P(z)/Q(z)$ and $\deg Q \geq \deg P + 2$. The function f has 6 simple poles in $\mathbb{C}$, and three of them are in the upper half-plane. These are located at:

$$z_1 = e^{i\pi/6}, \qquad z_2 = e^{i\pi/2}, \qquad z_3 = e^{i5\pi/6}.$$

The residues at these points are given by,

$$2\pi i \text{ Res}(f; z_1) = \frac{2\pi i}{6z_1^5} = \frac{\pi}{3} e^{-\pi i/3},$$
$$2\pi i \text{ Res}(f; z_2) = \frac{2\pi i}{6z_2^5} = \frac{\pi}{3},$$
$$2\pi i \text{ Res}(f; z_3) = \frac{2\pi i}{6z_3^5} = \frac{\pi}{3} e^{\pi i/3}$$

Therefore, by (1):

$$\text{PV} \int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = 2\pi i \sum_{j=1}^{3} \text{Res}(f; z_j) = \frac{\pi}{3}\left(1 + 2\cos\frac{\pi}{3}\right) = \frac{2\pi}{3},$$

and therefore,

$$\int_0^{\infty} \frac{1}{x^6 + 1} \, \mathrm{d}x = \frac{\pi}{3}$$

4.2.2. Evaluate the following real integrals by residue integration:
(a) $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} \, \mathrm{d}x, \quad a^2 > 0$
(d) $\int_0^{\infty} \frac{\cos kx}{x^4 + 1} \, \mathrm{d}x, \quad k$ real
(g) $\int_0^{\pi/2} \sin^4 \theta \, \mathrm{d}\theta$

Solution:
(a) Define I as the integral we seek to compute. Then

$$I = \text{Im}\,(J)\,, \qquad\qquad J \coloneqq \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + a^2} \, \mathrm{d}x,$$

and we will compute J to determine I. Define $f(z)$ as the rational part of the integrand for J:

$$f(z) \coloneqq \frac{z}{z^2 + a^2}.$$

Let C_R be the circular contour that is the portion of $\partial B_R(0)$ in the upper half-plane. Then $[-R, R]$ unioned with C_R is a closed contour. By the Cauchy Residue Theorem,

$$\lim_{R \uparrow \infty} \int_{C_R} f(z) e^{iz} \, \mathrm{d}z + \text{PV} \int_{-\infty}^{\infty} e^{iz} f(z) \, \mathrm{d}z = 2\pi i \sum_{j=1}^{M} \text{Res}(f(z) e^{iz}; z_j), \tag{2}$$

where $\{z_j\}_{j=1}^M$ the singularities of f in the upper half-plane. The function f has one such lone singularity ($M = 1$) at $z = i|a|$, with residue,

$$2\pi i \text{Res}(f(z) e^{iz}; i|a|) = 2\pi i \frac{i|a| e^{-|a|}}{2i|a|} = e^{-|a|} \pi i.$$

Note that

$$\lim_{R \uparrow \infty} \max_{z \in C_R} |f(z)| \leq \lim_{R \uparrow \infty} \frac{1}{R} = 0,$$

and so f uniformly decays to 0 on C_R as $R \uparrow \infty$. Thus, by Jordan's Lemma,

$$\lim_{R\uparrow\infty} \int_{C_R} f(z) e^{iz} \, \mathrm{d}z = 0.$$

Putting all this together in (2), we have,

$$J = \mathrm{PV} \int_{-\infty}^{\infty} e^{iz} f(z) \, \mathrm{d}z = i\pi e^{-|a|},$$

and therefore,

$$I = \mathrm{Im}\,(J) = \pi e^{-|a|}.$$

(d) We use a similar technique as in part (a). With I the integral we seek to compute, then

$$I = \frac{1}{2}\mathrm{Re}\,(J)\,, \qquad\qquad J := \int_{-\infty}^{\infty} e^{i|k|x} x^4 + 1 \, \mathrm{d}x,$$

where we have used the fact that the integrand for I is an even function and is invariant under $k \leftarrow |k|$. Then with C_R as in part (a), the Cauchy Residue Theorem implies,

$$\lim_{R\uparrow\infty} \int_{C_R} f(z) e^{i|k|z} \, \mathrm{d}z + J = 2\pi i \sum_{j=1}^{M} \mathrm{Res}(f(z)e^{i|k|z}; z_j), \tag{3}$$

where

$$f(z) := \frac{1}{z^4+1},$$

and $\{z_j\}_{j=1}^M$ are the singularities of f in the upper half-plane. We again have that f decays uniformly to 0 as $R \uparrow \infty$:

$$\lim_{R\uparrow\infty} \max_{z\in C_R} |f(z)| \le \lim_{R\uparrow\infty} \frac{1}{R^4-1} = 0,$$

and so by Jordan's Lemma,

$$\lim_{R\uparrow\infty} \int_{C_R} f(z) e^{i|k|z} \, \mathrm{d}z = 0, \quad |k| > 0.$$

The same result is true if $k = 0$ since f is a rational function $f = P/Q$ with $\deg Q \ge \deg P + 2$, i.e., we have

$$\lim_{R\uparrow\infty} \int_{C_R} f(z) e^{i|k|z} \, \mathrm{d}z = 0, \quad |k| \ge 0.$$

There are $M = 2$ singularities of $f(z)e^{i|k|z}$ in the upper half-plane located at $z_1 = e^{i\pi/4}$ and $z_2 = e^{3i\pi/4}$, with residues given by,

$$2\pi i \; \mathrm{Res}(f(z)e^{i|k|z}; z_1) = 2\pi i \frac{e^{i|k|z_1}}{4z_1^3} = -\frac{i\pi z_1}{2} e^{i|k|z_1},$$

$$2\pi i \; \mathrm{Res}(f(z)e^{i|k|z}; z_2) = 2\pi i \frac{e^{i|k|z_2}}{4z_2^3} = -\frac{i\pi z_3}{2} e^{i|k|z_3}.$$

so that (3) becomes,

$$\begin{aligned}
J &= -\frac{\pi}{2}\left(iz_1 e^{i|k|z_1} + iz_3 e^{i|k|z_3}\right) \\
&= -\frac{\pi e^{-|k|/\sqrt{2}}}{2}\left(iz_1 e^{i|k|/\sqrt{2}} + iz_3 e^{-i|k|/\sqrt{2}}\right) \\
&= -\frac{\pi e^{-|k|/\sqrt{2}}}{2}\left(e^{i\left(3\pi/4+|k|/\sqrt{2}\right)} + e^{i\left(5\pi/4-|k|/\sqrt{2}\right)}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
I = \frac{1}{2}\mathrm{Re}\,(J) &= -\frac{\pi e^{-|k|/\sqrt{2}}}{4}\left(\cos\left(\frac{3\pi}{4}+\frac{|k|}{\sqrt{2}}\right) + \cos\left(\frac{5\pi}{4}-\frac{|k|}{\sqrt{2}}\right)\right) \\
&= \frac{\pi e^{-|k|/\sqrt{2}}}{2\sqrt{2}}\left(\cos\frac{|k|}{\sqrt{2}} + \sin\frac{|k|}{\sqrt{2}}\right)
\end{aligned}$$

(g) Since $\sin^4\theta$ has period $\pi/2$, then

$$I = \int_0^{\pi/2} \sin^4\theta \,\mathrm{d}\theta = \frac{1}{4}\int_0^{2\pi} \sin^4\theta\,\mathrm{d}\theta.$$

We now use the parameterization $z = e^{i\theta}$, so that $\sin\theta = \frac{1}{2i}(z - 1/z)$, yielding,

$$\int_0^{2\pi}\sin^4\theta\,\mathrm{d}\theta = \int_{\partial B_1(0)} \frac{(z^2-1)^4}{16z^5}\frac{\mathrm{d}z}{i}.$$

We compute this latter integral via the Cauchy Residue Theorem:

$$\begin{aligned}
\int_{\partial B_1(0)} \frac{(z^2-1)^4}{16z^5}\frac{\mathrm{d}z}{i} &= \frac{2\pi i}{16i}\,\mathrm{Res}\left(\frac{(z^2-1)^4}{z^5};0\right) \\
&= \frac{\pi}{8}\frac{1}{4!}\left(\frac{\mathrm{d}^4}{\mathrm{d}z^4}(z^2-1)^4\right)\Big|_{z=0} \\
&= \frac{\pi}{8(4!)}\frac{\mathrm{d}^4}{\mathrm{d}z^4}\left(z^8 - 4z^6 + 6z^4 - 4z^2 + 1\right)\Big|_{z=0} = \frac{3\pi}{4}.
\end{aligned}$$

Thus,

$$I = \frac{1}{4}\frac{3\pi}{4} = \frac{3\pi}{16}$$

4.2.5. Consider a rectangular contour with corners at $b \pm iR$ and $b + 1 \pm iR$. Use this contour to show that,

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{b-iR}^{b+iR}\frac{e^{az}}{\sin\pi z}\,\mathrm{d}z = \frac{1}{\pi(1+e^{-a})},$$

where $0 < b < 1$ and $|\mathrm{Im}\,(a)| < \pi$.

Solution: For finite R, let the left, right, bottom, and top sides of the rectange be denote C_ℓ, C_r, C_b, and C_t, respectively. The integrand has singularities at $z = n$, $n \in \mathbb{Z}$, which are all simple poles, but $z = 1$ is the only singularity lying inside this contour. Therefore,

$$\text{Res}\left(\frac{e^{az}}{\sin \pi z}; 1\right) = \frac{e^a}{\pi \cos \pi} = -\frac{e^a}{\pi}.$$

Letting,

$$f(z) = \frac{e^{az}}{\sin \pi z},$$

then

$$\begin{aligned}
\left|\int_{C_b} f(z)\,\mathrm{d}z\right| &= \left|\int_b^{b+1} \frac{e^{a(x-iR)}}{\sin \pi(x-iR)}\,\mathrm{d}x\right| \\
&\leq \int_b^{b+1} \left| 2i \frac{e^{x\mathrm{Re}(a)+R\mathrm{Im}(a)} e^{i(x\mathrm{Im}(a)-R\mathrm{Re}(a))}}{e^{i\pi x+\pi R} - e^{-\pi R - i\pi x}} \right| \mathrm{d}x \\
&= 2e^{R\mathrm{Im}(a)} \int_b^{b+1} \frac{e^{x\mathrm{Re}(a)}}{\left|e^{i\pi x+\pi R} - e^{-\pi R-i\pi x}\right|}\,\mathrm{d}x \\
&\leq \frac{2e^{R\mathrm{Im}(a)}}{e^{\pi R}} \frac{1}{1-e^{-2\pi R}} \int_b^{b+1} e^{x\mathrm{Re}(a)}\,\mathrm{d}x \\
&\leq 2e^{R(\mathrm{Im}(a)-\pi)} \frac{\max\{e^{b\mathrm{Re}(a)}, e^{(b+1)\mathrm{Re}(a)}\}}{1-e^{-2\pi R}}.
\end{aligned}$$

Therefore, taking the limit in R and noting that $\mathrm{Im}\,(a) - \pi < 0$, then,

$$\lim_{R\uparrow\infty} \int_{C_b} f(z)\,\mathrm{d}z = 0.$$

A similar computation can be carried out for C_t by simply performing the same computation as on C_b but by making the replacement $R \leftarrow -R$:

$$\begin{aligned}
\left|\int_{C_t} f(z)\,\mathrm{d}z\right| &\leq 2e^{-R\mathrm{Im}(a)} \int_b^{b+1} \frac{e^{x\mathrm{Re}(a)}}{\left|e^{i\pi x-\pi R} - e^{\pi R-i\pi x}\right|}\,\mathrm{d}x \\
&\leq 2\frac{e^{-R\mathrm{Im}(a)}}{e^{\pi R}} \frac{1}{1-e^{-2\pi R}} \int_b^{b+1} e^{x\mathrm{Re}(a)}\,\mathrm{d}x \\
&\leq 2e^{R(-\mathrm{Im}(a)-\pi)} \frac{\max\{e^{b\mathrm{Re}(a)}, e^{(b+1)\mathrm{Re}(a)}\}}{1-e^{-2\pi R}}
\end{aligned}$$

We also have $-\mathrm{Im}\,(a) - \pi < 0$, so taking limits in R yields:

$$\lim_{R\uparrow\infty} \int_{C_t} f(z)\,\mathrm{d}z = 0.$$

On the left contour, C_ℓ, we have,

$$\begin{aligned}\int_{C_\ell} f(z) &= \int_{b+iR}^{b-iR} f(z)\,\mathrm{d}z \\ &= \int_{R}^{-R} f(b+iy) i\,\mathrm{d}y \\ &= -\int_{-R}^{R} f(b+iy) i\,\mathrm{d}y \\ &= -\int_{b-iR}^{b+iR} f(z)\,\mathrm{d}z =: -2\pi i\, I(R),\end{aligned}$$

i.e., we have defined

$$I(R) \coloneqq \frac{1}{2\pi i}\int_{b-iR}^{b+iR} \frac{e^{az}}{\sin \pi z}\,\mathrm{d}z.$$

On the right contour C_r we have,

$$\begin{aligned}\int_{C_r} f(z) &= \int_{b+1-iR}^{b+1+iR} \frac{e^{az}}{\sin \pi z}\,\mathrm{d}z \\ &\overset{w=z-1}{=} e^a \int_{b-iR}^{b+iR} \frac{e^{aw}}{\sin(\pi w - \pi)}\,\mathrm{d}w \\ &= -e^a \int_{b-iR}^{b+iR} \frac{e^{aw}}{\sin \pi w}\,\mathrm{d}w = -e^a (2\pi i) I(R).\end{aligned}$$

Then the Cauchy Residue Theorem states,

$$\int_{C_b} f(z)\,\mathrm{d}z + \int_{C_t} f(z)\,\mathrm{d}z + \int_{C_\ell} f(z)\,\mathrm{d}z + \int_{C_r} f(z)\,\mathrm{d}z = 2\pi i \mathrm{Res}(f;1),$$

and using all our computations above yields,

$$\int_{C_b} f(z)\,\mathrm{d}z + \int_{C_t} f(z)\,\mathrm{d}z + 2\pi i \left(-1 - e^a\right) I(R) = -2\pi i \frac{e^a}{\pi}$$

Taking limits in R:

$$\lim_{R\uparrow\infty} I(R) = \frac{e^a}{\pi(1+e^a)} = \frac{1}{\pi(1+e^{-a})},$$

which is what we wished to show.

4.2.7. Use a sector contour with radius R, as in Figure 4.2.6 in the text, centered at the origin with angle $0 \le \theta \le \frac{2\pi}{5}$ to find, for $a > 0$,

$$\int_0^\infty \frac{\mathrm{d}x}{x^5 + a^5} = \frac{\pi}{5a^4 \sin \frac{\pi}{5}}.$$

Solution: We use the Cauchy Residue Theorem, and so proceed to define and integrate along a closed contour. The contour we consider contains two rays of length R, one extending from the origin at angle 0, and the second extending from the origin at angle $\frac{2\pi}{5}$. We denote these two contours by C_0 (angle 0) and C_+ (angle $2\pi/5$), respectively. We will call the circular arc of radius R connecting these as C_R. Defining,

$$f(z) = \frac{1}{z^5 + a^5},$$

which satisfies,

$$\lim_{R\to\infty} \max_{z\in C_R} |zf(z)| = \lim_{R\to\infty} \max_{z\in C_R} \frac{R}{|z^5 + a^5|} \leq \lim_{R\to\infty} \max_{z\in C_R} \frac{R}{R^5 - a^5} = 0,$$

then we have,

$$\lim_{R\to\infty} \int_{C_R} f(z)\,\mathrm{d}z = 0.$$

Along the contour C_0, through the parameterization $z = x$ as x ranges from 0 to R, we have,

$$\lim_{R\to\infty} \int_{C_0} f(z)\,\mathrm{d}z = \int_0^\infty \frac{\mathrm{d}x}{x^5 + a^5} =: I.$$

Along the contour C_+, through the parameterization $z = re^{2\pi i/5}$, as r ranges from R to 0, we have,

$$\lim_{R\to\infty} \int_{C_+} f(z)\,\mathrm{d}z = \int_\infty^0 \frac{e^{2\pi i/5}\,\mathrm{d}r}{r^5 + a^5} = -e^{2\pi i/5} I$$

Finally, the singularities of f are all simple poles at the points,

$$z = z_j := a^{1/5} e^{i\pi/5} e^{i2\pi j/5}, \qquad\qquad j = 0, 1, 2, 3, 4,$$

and only one of these poles, z_0, lies inside the contour. Its corresponding residue is,

$$2\pi i \,\mathrm{Res}(f; z_0) = \frac{2\pi i}{5 z_0^4} = \frac{2\pi i}{5a^4} e^{-4\pi i/5}$$

Finally, the Cauchy Residue Theorem integrating over C_0, C_R, and C_+, after taking the limit $R \to \infty$, reads,

$$I + 0 - e^{2\pi i/5} I = \frac{\pi}{5a^4} 2i e^{-4\pi i/5}$$

Rearranging, this yields,

$$\begin{aligned} I &= \frac{\pi}{5a^4} \frac{2ie^{-4\pi i/5}}{e^{i\pi/5}\left(e^{-i\pi/5} - e^{i\pi/5}\right)} \\ &= \frac{\pi}{5a^4} \frac{-2i}{e^{-i\pi/5} - e^{i\pi/5}} \\ &= \frac{\pi}{5a^4} \frac{1}{\sin\frac{\pi}{5}}, \end{aligned}$$

which is what we wanted to show.

4.3.2. Show that,

$$\int_0^\infty \frac{\sin x}{x(x^2+1)}\,\mathrm{d}x = \frac{\pi}{2}\left(1 - \frac{1}{e}\right).$$

Solution: We start by defining,

$$f(z) = \frac{e^{iz}}{z(z^2+1)}.$$

Then we have,

$$J := \int_0^\infty f(z)\,\mathrm{d}z, \qquad\qquad I = \mathrm{Im}\,(J)\,,$$

where I is the integral we seek to compute. We will evaluate J using the Cauchy Residue Theorem, with a closed loop consisting of (i) a radius R semicircular contour C_R centered at 0 in the upper half-place with large R, (ii) the integral along the real interval $I_- = (-R, -\epsilon)$ for $\epsilon > 0$ small, (iii) the semicirular contour C_ϵ in the upper half plane centered at 0, (iv) the integral along the real interval $I_+ = (\epsilon, R)$. We will take limits as $R \uparrow \infty$ and $\epsilon \downarrow 0$. We proceed to compute these integrals.
First, we have that,

$$\lim_{\epsilon\to 0}\int_{C_\epsilon} f(z)\,\mathrm{d}z = -i\pi\mathrm{Res}(f;0) = -i\pi\frac{e^0}{(0^2+1)} = -i\pi.$$

since C_ϵ sweeps out an angle of π with clockwise orientation. For $|z| = R > 1$, we have,

$$\left|\frac{1}{z(z^2+1)}\right| \leq \frac{1}{R(R^2-1)} \overset{R\to\infty}{\to} 0,$$

and hence by Jordan's Lemma,

$$\lim_{R\to\infty}\int_{C_R} f(z)\,\mathrm{d}z = \lim_{R\to\infty}\int_{C_R}\frac{e^{iz}}{z(z^2+1)}\,\mathrm{d}z = 0.$$

We next compute the two integrals on the real line:

$$\lim_{\epsilon\to 0^+, R\to\infty}\int_{I_+} f(z)\,\mathrm{d}z = J,$$

and

$$\lim_{\epsilon\to 0^+, R\to\infty}\int_{I_-} f(z)\,\mathrm{d}z = \lim_{\epsilon\to 0^+, R\to\infty} -\int_\epsilon^\infty \frac{e^{-ix}}{x(x^2+1)}\,\mathrm{d}x = -\overline{J}.$$

Finally, the only residue of f in the upper half plane is located at $z = i$:

$$2\pi i\mathrm{Res}(f;i) = 2\pi i\frac{e^{-1}}{i(2i)} = -\frac{i\pi}{e}$$

Finally, the Cauchy Residue Theorem yields:

$$\lim_{\epsilon\to 0^+, R\to\infty}\left[\int_{C_R} f(z)\,\mathrm{d}z + \int_{I_-} f(z)\,\mathrm{d}z + \int_{C_\epsilon} f(z)\,\mathrm{d}z + \int_{I_+} f(z)\,\mathrm{d}z\right] = 2\pi i \mathrm{Res}(f;i),$$

i.e.,

$$-i\pi + J - \overline{J} = -\frac{i\pi}{e} \quad\Longrightarrow\quad I = \mathrm{Im}\,(J) = \frac{1}{2}\left(\pi - \frac{\pi}{e}\right) = \frac{\pi}{2}\left(1-\frac{1}{e}\right)$$

4.3.3. Show that,

$$\int_{-\infty}^{\infty} \frac{\cos x - 1}{x^2(x^2+a^2)}\,\mathrm{d}x = -\frac{\pi}{a^2} + \frac{\pi}{a^3}\left(1-e^{-a}\right), \quad a>0.$$

Solution: We use the same contour as in the previous problem's solution (4.3.2), in particular with the curves $I_\pm$, C_ϵ, and C_R. We define,

$$f(z) = \frac{e^{iz}-1}{z^2(z^2+a^2)},$$

whose single residue in the upper half-plane is at $z = ia$:

$$2\pi i\mathrm{Res}(f;ia) = 2\pi i \frac{e^{-a}-1}{-a^2(2ia)} = \pi\frac{1-e^{-a}}{a^3}.$$

To evaluate along C_R, we note that for $|z| = R$,

$$\left|\frac{1}{z^2+(z^2+a^2)}\right| \leq \frac{1}{R^2(R^2-a^2)} \stackrel{R\to\infty}{\to} 0,$$

and hence by a combination of Jordan's Lemma, and the result that the integral along C_R of a rational function $P(z)/Q(z)$ with $\deg Q(z) \geq \deg P(z) + 2$ goes to 0, we have,

$$\lim_{R\to\infty}\int_{C_R} f(z)\,\mathrm{d}z = 0.$$

To evaluate along C_ϵ, first note that,

$$f(z) = \frac{e^{iz}-1}{z^2(z^2+a^2)} = \frac{1}{z^2+a^2}\left(\frac{i}{z} - \frac{1}{2} + \dots\right),$$

and hence f as a simple pole at $z = 0$ with $\mathrm{Res}(f;0) = i/a^2$. Then,

$$\lim_{\epsilon\to 0}\int_{C_\epsilon} f(z)\,\mathrm{d}z = -i\pi\mathrm{Res}(f;0) = \pi/a^2.$$

On the intervals $I_\pm$ we have, after taking limits:

$$\int_{I_+} f(z)\,\mathrm{d}z \to \int_0^\infty \frac{e^{ix}-1}{x^2(x^2+a^2)}\,\mathrm{d}x$$
$$\int_{I_-} f(z)\,\mathrm{d}z \to \int_{-\infty}^0 \frac{e^{ix}-1}{x^2(x^2+a^2)}\,\mathrm{d}x = \int_0^\infty \frac{e^{-ix}-1}{x^2(x^2+a^2)}\,\mathrm{d}x$$

Finally, putting things together with the Cauchy Residue Theorem yields,

$$\int_0^\infty 2\frac{\cos x - 1}{x^2(x^2+a^2)}\,\mathrm{d}x + \frac{\pi}{a^2} = \pi\frac{1-e^{-a}}{a^3},$$

and using the fact that the integrand above is even, this implies

$$\int_{-\infty}^\infty \frac{\cos x - 1}{x^2(x^2+a^2)}\,\mathrm{d}x = -\frac{\pi}{a^2} + \frac{\pi}{a^3}\left(1-e^{-a}\right)$$

4.3.7. Use the keyhole contour of Figure 4.3.6 in the text to show that on the principal branch of x^k,
(a)

$$I(a) = \int_0^\infty \frac{x^{k-1}}{(x+a)}\,\mathrm{d}x = \frac{\pi}{\sin k\pi}a^{k-1}, \quad 0<k<1, \quad a>0$$

Solution: We use the same notation as in the figure: C_ϵ denotes a circle of radius $\epsilon > 0$ traversed clockwise with a small opening at $\arg z = 0$, and C_R denotes a circle of radius $R \gg 1$ with a small opening at $\arg z = 0$ traversed counterclockwise. We let I_+ denote the integral along $[\epsilon, R]$ with small positive imaginary part, and I_- the same integral but small negative imaginary part. Define

$$f(z) = \frac{z^{k-1}}{z+a}.$$

We begin by computing the (single) residue inside the contour at $z = -a$:

$$2\pi i \mathrm{Res}(f; -a) = 2\pi i (a e^{i\pi})^{k-1} = -2\pi i a^{k-1} e^{i\pi k}.$$

On the contour C_R with $|z| = R$, we have,

$$|zf(z)| \leq \frac{RR^{k-1}}{R-a} = R^{k-1}\frac{1}{1-a/R} \overset{R\to\infty}{\to} 0,$$

where we have used $k-1 \in (-1, 0)$ since $0<k<1$. Since this limit is uniform in z, then

$$\lim_{R\to\infty}\int_{C_R} f(z)\,\mathrm{d}z = 0.$$

A similar computation can be carried out on C_ϵ, where $|z| = \epsilon$ and $\epsilon \ll 1$:

$$|zf(z)| \leq \frac{\epsilon^k}{a-\epsilon} \overset{\epsilon\to 0^+}{\to} 0,$$

which holds uniformly in z where again we have used $0<k<1$. To understand the integrals on $I_\pm$, we define the branch of the function z^k to be so that $\arg z \in [0, 2\pi)$. The integral along I_+ is given via the parameterization $z = x$,

$$\int_\epsilon^R \frac{x^{k-1}}{x-a}\,\mathrm{d}x \to I(a),$$

and along I_- we use the parameterization $z = xe^{2\pi i}$ to yield[1],

$$\int_R^\epsilon \frac{x^{k-1}e^{2\pi i(k-1)}}{xe^{2\pi i} - a} e^{2\pi i}\,\mathrm{d}x = -e^{2\pi i k} \int_\epsilon^R \frac{x^{k-1}}{x+a}\,\mathrm{d}x \to -e^{2\pi i k} I(a).$$

Putting everything together with the Cauchy Residue Theorem yields,

$$I(a) = -2\pi i a^{k-1} e^{i\pi k} \frac{1}{1 - e^{2\pi i k}} = \pi a^{k-1} \frac{2i}{e^{i\pi k} - e^{-i\pi k}} = \frac{\pi}{\sin k\pi} a^{k-1}$$

4.3.13. Use the keyhole contour of Figure 4.3.6 to show for the principal branch of $x^{1/2}$ and $\log x$,

$$\int_0^\infty \frac{x^{1/2}\log x}{(1+x^2)}\,\mathrm{d}x = \frac{\pi^2}{2\sqrt{2}}$$

Solution: We use the same notation for the keyhole contour as in the solution to the previous problem (4.3.7), in particular for the contours C_R, C_ϵ, and $I_\pm$. Define,

$$f(z) = \frac{z^{1/2}\log z}{1+z^2},$$

where for both $z^{1/2}$ and $\log z$ we define our branch as that associated to $\arg z \in [0, 2\pi)$. This function has two residues inside the keyhole contour located at $z = \pm i$:

$$\begin{aligned} 2\pi i \mathrm{Res}(f; i) &= 2\pi i \frac{i^{1/2}\log i}{2i} = 2\pi i \frac{e^{i\pi/4} i \frac{\pi}{2}}{2i} = i\frac{\pi^2}{2} e^{i\pi/4} \\ 2\pi i \mathrm{Res}(f; -i) &= 2\pi i \frac{(-i)^{1/2}\log(-i)}{-2i} = 2\pi i \frac{e^{i3\pi/4} i \frac{3\pi}{2}}{-2i} = -i\frac{3\pi^2}{2} e^{i3\pi/4}, \end{aligned}$$

so that,

$$2\pi i \left(\mathrm{Res}(f; i) + \mathrm{Res}(f; -i)\right) = \frac{\pi^2}{2} e^{i\pi/4}(3+i) = \pi^2\left(\frac{1}{\sqrt{2}} + i\sqrt{2}\right).$$

On the contour C_R, we note that for $|z| = R$ with $R > 1$:

$$|zf(z)| = \frac{|z|^{3/2}|\log z|}{|z^2+1|} \leq \frac{R^{3/2}(\log R + 2\pi)}{R^2 - 1} \overset{R\uparrow\infty}{\to} 0,$$

uniformly for $z \in C_R$, which implies,

$$\lim_{R\to\infty} \int_{C_R} f(z)\,\mathrm{d}z = 0.$$

Similarly on C_ϵ, for $|z| = \epsilon$ and $\epsilon < 1$, we have,

$$|zf(z)| = \frac{|z|^{3/2}|\log z|}{|z^2+1|} \leq \frac{\epsilon^{3/2}(\log \epsilon + 2\pi)}{1 - \epsilon^2} \overset{\epsilon\downarrow 0}{\to} 0,$$

[1] Techically, the parameterization is $z = xe^{i(2\pi - \delta)}$ for infinitesimal $\delta > 0$.

again uniformly for $z \in C_\epsilon$, and therefore,

$$\lim_{\epsilon \to \infty} \int_{C_\epsilon} f(z)\,\mathrm{d}z = 0.$$

We can now compute the integrals on the contours $I_\pm$. On I_+, we use the parameterization $z = x$ with x real to obtain,

$$\int_{I_+} f(z)\,\mathrm{d}z = \int_\epsilon^R \frac{x^{1/2}\log x}{x^2+1}\,\mathrm{d}x \xrightarrow{R\uparrow\infty,\epsilon\downarrow 0} \int_0^\infty \frac{x^{1/2}\log x}{x^2+1}\,\mathrm{d}x =: J.$$

On I_-, we use the parameterization $z = xe^{2\pi i}$ to yield,

$$\begin{aligned}\int_{I_-} f(z)\,\mathrm{d}z &= \int_R^\epsilon \frac{(xe^{2\pi i})^{1/2}\log xe^{2\pi i}}{1+(xe^{2\pi i})^2} e^{2\pi i}\,\mathrm{d}x \\ &= \int_\epsilon^R \frac{x^{1/2}\left(\log x + 2\pi i\right)}{1+x^2}\,\mathrm{d}x \\ &\xrightarrow{R\uparrow\infty,\epsilon\downarrow 0} J + 2\pi i \int_0^\infty \frac{x^{1/2}}{1+x^2}\,\mathrm{d}x.\end{aligned}$$

Combining all this with the Cauchy Residue Theorem (and taking limits) yields,

$$2J + 2\pi i \int_0^\infty \frac{x^{1/2}}{x^2+1}\,\mathrm{d}x = \frac{\pi^2}{\sqrt{2}} + i\sqrt{2}\pi^2,$$

and taking real parts of the above equality implies,

$$\int_0^\infty \frac{x^{1/2}\log x}{(1+x^2)}\,\mathrm{d}x = J = \frac{\pi^2}{2\sqrt{2}}$$