Department of Mathematics, University of Utah

Applied Complex Variables and Asymptotic Methods
MATH 6720 – Section 001 – Spring 2024
Homework 5 Solutions
Residue Calculus, I

Due: Friday, March 15, 2024

Below, problem C in section A.B is referred to as exercise A.B.C.

Text: *Complex Variables: Introduction and Applications*, Ablowitz & Fokas,

Exercises: 4.1.1, parts (b), (d), and (e)
4.1.2, parts (b) and (c)
4.1.8

Submit your homework assignment on Canvas via Gradescope.

4.1.1. Evaluate the integrals $\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z$, where C is the unit circle centered at the origin and $f(z)$ is given below.

(b) $\frac{\cosh(1/z)}{z}$
(d) $\frac{\log(z+2)}{2z+1}$, principal branch
(e) $\frac{z+1/z}{z(2z-1/2z)}$

Solution: We will evaluate these integrals using the Cauchy Residue Theorem.

(b) The only singularity inside C is at $z = 0$, which is an essential singularity due to the Laurent expansion of $\cosh(1/z)$. Therefore, we compute the residue using the Laurent expansion of f around 0:

$$f(z) = \frac{1}{z}\left(\sum_{n=0}^{\infty} \frac{1}{z^{2n}(2n)!}\right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z^{2n+1}(2n)!},$$

and therefore $\mathrm{Res}(f;0) = 1$, so the Residue Theorem implies that

$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \mathrm{Res}(f;0) = 1.$$

(d) For the principal branch of $w \mapsto \log w$, we take $w = re^{i\theta}$ with $\theta \in [-\pi,\pi)$. The singularity of the numerator $\log(z+2)$ lies at $z = -2$, which is outside C, and the denominator has a simple zero at $z = -1/2$, which lies inside C. Therefore, we need only compute $\mathrm{Res}(f;-1/2)$. This can be directly computed as,

$$\mathrm{Res}(f;-1/2) = \frac{\log(-1/2+2)}{(2z+1)'|_{z=-1/2}} = \frac{1}{2}\log\frac{3}{2}.$$

By the Residue Theorem,

$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \mathrm{Res}(f;-1/2) = \frac{1}{2}\log\frac{3}{2}.$$

(e) We rewrite this function as,

$$f(z) = \frac{2z^2 + 2}{z(4z^2 - 1)}.$$

Since the denominator is a polynomial with simple zeros, then f has simple poles at $z = 0$ and $z = \pm 1/2$, all of which lie inside C. We compute the corresponding residues as follows:

$$\begin{aligned}
\text{Res}(f; 0) &= \frac{(2z^2+2)\big|_{z=0}}{(z(4z^2-1))'\big|_{z=0}} = \frac{2}{-1} = -2, \\
\text{Res}(f; +1/2) &= \frac{(2z^2+2)\big|_{z=1/2}}{(z(4z^2-1))'\big|_{z=1/2}} = \frac{5/2}{z(8z)\big|_{z=1/2}} = \frac{5}{4}, \\
\text{Res}(f; -1/2) &= \frac{(2z^2+2)\big|_{z=-1/2}}{(z(4z^2-1))'\big|_{z=-1/2}} = \frac{5/2}{z(8z)\big|_{z=-1/2}} = \frac{5}{4}.
\end{aligned}$$

Then by the Residue Theorem,

$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \text{Res}(f;0) + \text{Res}(f;1/2) + \text{Res}(f;-1/2) = \frac{1}{2}.$$

4.1.2. Evaluate the integrals $\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z$, where C is the unit circle centered at the origin with $f(z)$ given below. Do these problems by both (i) enclosing the singular points inside C and (ii) enclosing the singular points outside C (by including the point at infinity). Show that you obtain the same result in both cases.

(b) $\frac{z^2+1}{z^3}$

(c) $z^2 e^{-1/z}$

Solution: We will evaluate these integrals using the Cauchy Residue Theorem using two different ways. First we note that as a consequence of the definition,

$$\text{Res}(f;\infty) := \lim_{R\to\infty} \frac{1}{2\pi i}\oint_{\partial B_R(0)} f(z)\,\mathrm{d}z,$$

then $\sum_{j=1}^{M} \text{Res}(f; z_j) = \text{Res}(f;\infty)$, where $\{z_j\}_{j=1}^{M}$ are the singularities of f in the finite plane $\mathbb{C}$.

(b) The only singularity inside C is at $z = 0$, and there are no singularities in the finite plane outside C. Therefore,

$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \text{Res}(f;0) = \text{Res}(f;\infty).$$

The Laurent expansion of f at 0 is given by the function itself, $f(z) = \frac{1}{z} + \frac{1}{z^3}$, so $\text{Res}(f;0) = 1$. To compute the Residue at infinity, we use the formula,

$$\text{Res}(f(z);\infty) = \text{Res}\left(\frac{1}{w^2} f\left(\frac{1}{w}\right); 0\right) = \text{Res}\left(\frac{1}{w} + w; 0\right) = 1.$$

Hence we have,

$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \mathrm{Res}(f;\infty) = 1,$$
$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \mathrm{Res}(f;0) = 1,$$

as expected.

(c) Again, the only singularity inside C is at $z = 0$, and there are no singularities in the finite plane outside C. So again we have,

$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \mathrm{Res}(f;0) = \mathrm{Res}(f;\infty).$$

The point $z = 0$ is an essential singularity for f, so we compute the Laurent expansion:

$$f(z) = z^2\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n n!} = z^2 - z + \frac{1}{2} - \frac{1}{6z} + \ldots,$$

so $\mathrm{Res}(f;0) = -1/6$. The residue at infinity is given by,

$$\mathrm{Res}(f(z);\infty) = \mathrm{Res}\left(\frac{1}{w^2}f\left(\frac{1}{w}\right);0\right) = \mathrm{Res}\left(\frac{e^{-w}}{w^4};0\right) = -\frac{1}{6}.$$

Therefore, we have,

$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \mathrm{Res}(f;\infty) = -\frac{1}{6},$$
$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \mathrm{Res}(f;0) = -\frac{1}{6}.$$

4.1.8. Suppose $f(z)$ is a meromorphic function (i.e., $f(z)$ is analytic everywhere in the finite z plane except at isolated points where it has poles) with N simple zeros (i.e., $f(z_0) = 0$, $f'(z_0) \neq 0$) and M simple poles inside a circle C. Show that,

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = N - M.$$

Solution: Note that $\frac{f'(z)}{f(z)}$ has singularities only where either $f(z)$ and/or $f'(z)$ have singularities, or where $f(z)$ has zeros. Since $f'(z)$ is analytic inside C everywhere that f is analytic, then the only singularities of $\frac{f'(z)}{f(z)}$ occur where f has singularities or zeros. Let $\{z_j\}_{j=1}^N$ be the zeros (simple) of f, and let $\{w_k\}_{k=1}^M$ be the simple poles of f. Then by the Residue Theorem,

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{j=1}^{N}\mathrm{Res}\left(\frac{f'(z)}{f(z)};z_j\right) + \sum_{k=1}^{M}\mathrm{Res}\left(\frac{f'(z)}{f(z)};w_k\right). \tag{1a}$$

For a fixed j, since z_j is a simple zero, then in a neighborhood of z_j we have,

$$f(z) = (z - z_j)g(z), \quad g(z) \neq 0,\ g(z) \text{ is analytic.}$$

In this neighborhood, we compute,

$$f'(z) = g(z) + (z - z_j)g'(z) \quad \Longrightarrow \quad \frac{f'(z)}{f(z)} = \frac{1}{z - z_j} + \frac{g'(z)}{g(z)},$$

with $g'(z)/g(z)$ analytic in this neighborhood since g has no zeros or singularities in this neighborhood. Therefore,

$$\operatorname{Res}\left(\frac{f'(z)}{f(z)}; z_j\right) = 1. \tag{1b}$$

Now consider for a fixed k a similar computation in a neighborhood of w_k, which is a simple pole:

$$f(z) = \frac{h(z)}{z - w_k}, \quad h(z) \neq 0, \;\; h(z) \text{ is analytic.}$$

Then in this neighborhood, we have

$$f'(z) = -\frac{h(z)}{(z - w_k)^2} + \frac{h'(z)}{z - w_k} \quad \Longrightarrow \quad \frac{f'(z)}{f(z)} = -\frac{1}{z - w_k} + \frac{h'(z)}{h(z)},$$

and $h'(z)/h(z)$ is analytic in this neighborhod since h has no zeros or singularities in this neighborhood. Therefore,

$$\operatorname{Res}\left(\frac{f'(z)}{f(z)}; w_k\right) = -1. \tag{1c}$$

Combining the three equalities (1a), (1b), and (1c) proves the desired result.