Department of Mathematics, University of Utah

Applied Complex Variables and Asymptotic Methods

MATH 6720 – Section 001 – Spring 2024

Homework 3

Complex Integration, II

Due: Friday, Feb 16, 2024

Below, problem C in section A.B is referred to as exercise A.B.C.
Text: *Complex Variables: Introduction and Applications*, Ablowitz & Fokas,

Exercises: 2.4.8
2.5.2
2.5.3
2.6.2
2.6.5
2.6.7
3.2.3
Supplement 3.1

Submit your homework assignment on Canvas via Gradescope.

2.4.8. Let C be an arc of the circle $|z| = R$, $(R > 1)$ of angle $\pi/3$. Show that

$$\left| \int_C \frac{\mathrm{d}z}{z^3+1} \, \mathrm{d}z \right| \leq \frac{\pi}{3} \left(\frac{R}{R^3-1} \right),$$

and deduce $\lim_{R\to\infty} \int_C \frac{\mathrm{d}z}{z^3+1} \, \mathrm{d}z = 0$.

2.5.2. Use partial fractions to evaluate the following integrals $\oint_C f(z) \, \mathrm{d}z$, where C is the unit circle centered at the origin, and $f(z)$ is given by the following:

(a) $\frac{1}{z(z-2)}$
(b) $\frac{z}{z^2-1/9}$
(c) $\frac{1}{z(z+\frac{1}{2})(z-2)}$

2.5.3. Evaluate the following integral,

$$\oint_C \frac{e^{iz}}{z(z-\pi)} \, \mathrm{d}z,$$

for each of the following four cases (all circle are centered at the origin; use Eq. (1.2.19) as necessary).

(a) C is the boundary of the annulus between circles of radius 1 and radius 3.
(b) C is the boundary of the annulus between circles of radius 1 and radius 4.
(c) C is a circle of radius R, where $R > \pi$.
(d) C is a circle of radius R, where $R < \pi$.

2.6.2. Evaluate the integrals $\oint_C f(z)\,\mathrm{d}z$ over a contour C, where C is the boundary of a square with diagonal opposite corners at $z = -(1+i)R$ and $z = (1+i)R$, where $R > a > 0$, and where $f(z)$ is given by the following (use Eq. (1.2.19) in the text as necessary):

(a) $\frac{e^z}{z - \frac{\pi i}{4}a}$
(b) $\frac{e^z}{\left(z - \frac{\pi i}{4}a\right)^2}$
(c) $\frac{z^2}{2z+a}$
(d) $\frac{\sin z}{z^2}$
(e) $\frac{\cosh z}{z}$

2.6.5. Consider two entire functions with no zeros and having a ratio equal to unity at infinity. Use Liouville's Theorem to show that they are in fact the same function.

2.6.7. Let $f(z)$ be an entire function, with $|f(z)| \leq C|z|$ for all z, where C is a constant. Show that $f(z) = Az$, where A is a constant.

3.2.3. Let the Euler number E_n be defined by the power series,

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n.$$

(a) Find the radius of convergence of this series.
(b) Determine the first six Euler numbers.

Supplement 3.1. Prove the Cauchy Integral Formula: Let f be analytic in an open domain D, and let $z \in D$. Then for any non-negative integer n,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}}\,\mathrm{d}w$$

where C is any simple contour in D enclosing z. You may take the $n = 0, 1$ versions of this formula, proven in class, as given.