Department of Mathematics, University of Utah

Applied Complex Variables and Asymptotic Methods

MATH 6720 – Section 001 – Spring 2024

Homework 3 Solutions

Complex Integration, II

Due: Friday, Feb 16, 2024

Below, problem C in section A.B is referred to as exercise A.B.C.
Text: *Complex Variables: Introduction and Applications*, Ablowitz & Fokas,

Exercises: 2.4.8
2.5.2
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Supplement 3.1

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2.4.8. Let C be an arc of the circle $|z| = R$, $(R > 1)$ of angle $\pi/3$. Show that

$$\left| \int_C \frac{\mathrm{d}z}{z^3+1}\,\mathrm{d}z \right| \leq \frac{\pi}{3}\left(\frac{R}{R^3-1}\right), \tag{1}$$

and deduce $\lim_{R\to\infty} \int_C \frac{\mathrm{d}z}{z^3+1}\,\mathrm{d}z = 0$.

Solution: We seek to use the result stating that if $|f(z)| \leq M$ over a contour C of arclength L, then,

$$\left| \int_C f(z)\,\mathrm{d}z \right| \leq ML. \tag{2}$$

The length of this contour (a radius-R circular arc of angle $\pi/3$) has value $L = R\pi/3$. To compute M for $f(z) = \frac{1}{z^3+1}$, we note that

$$\max_{z\in C} |f(z)| \leq \max_{|z|=R} |f(z)| = \max_{|z|=R} \left|\frac{1}{z^3+1}\right| = \frac{1}{\min_{|z|=R} |z^3+1|}.$$

We compute the desired minimum via the (reverse) triangle inequality:

$$\min_{|z|=R} |z^3+1| \geq \min_{|z|=R} |z^3| - 1 = R^3 - 1.$$

Hence, we have

$$\max_{z\in C} |f(z)| \leq \frac{1}{R^3-1} := M.$$

Using $L = R\pi/3$ and $M = 1/(R^3 - 1)$ in (2) proves (1). The subsequent limit is immediate:

$$\lim_{R\to\infty} \int_C \frac{\mathrm{d}z}{z^3+1}\,\mathrm{d}z \le \lim_{R\to\infty} \left| \int_C \frac{\mathrm{d}z}{z^3+1}\,\mathrm{d}z \right| \le \lim_{R\to\infty} \frac{\pi}{3}\frac{R}{R^3-1} = 0.$$

2.5.2. Use partial fractions to evaluate the following integrals $\oint_C f(z)\,\mathrm{d}z$, where C is the unit circle centered at the origin, and $f(z)$ is given by the following:

(a) $\frac{1}{z(z-2)}$
(b) $\frac{z}{z^2-1/9}$
(c) $\frac{1}{z(z+\frac{1}{2})(z-2)}$

Solution:

(a) We know that for an arbitrary complex number a,

$$\oint_C \frac{1}{z-a}\,\mathrm{d}z = \begin{cases} 0, & a \text{ is outside } C \\ 2\pi i, & a \text{ is inside } C \end{cases} \tag{3}$$

We will use this property to evaluate the integral once we have expanded in partial fractions. $f(z)$ has poles at $z = 0, z = 2$, so we use the ansatz,

$$f(z) = \frac{C_1}{z} + \frac{C_2}{z-2}.$$

By clearing denominators, this leads to the following linear system for the unknowns C_1, C_2:

$$\left.\begin{array}{c} -2C_1 = 1, \\ C_1 + C_2 = 0 \end{array}\right\} \Longrightarrow (C_1, C_2) = \left(-\frac{1}{2}, \frac{1}{2}\right).$$

Hence,

$$\oint_C f(z)\,\mathrm{d}z = -\frac{1}{2}\oint_C \frac{1}{z}\,\mathrm{d}z + \frac{1}{2}\oint_C \frac{1}{z-2}\,\mathrm{d}z \overset{(3)}{=} -\pi i$$

(b) The partial fractions ansatz in this case is,

$$f(z) = \frac{C_1}{z-1/3} + \frac{C_2}{z+1/3},$$

resulting in the linear system,

$$\left.\begin{array}{c} C_1 + C_2 = 1, \\ C_1 - C_2 = 0 \end{array}\right\} \Longrightarrow (C_1, C_2) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Therefore,

$$\oint_C f(z)\,\mathrm{d}z = \frac{1}{2}\oint_C \frac{1}{z-1/3}\,\mathrm{d}z + \frac{1}{2}\oint_C \frac{1}{z+1/3}\,\mathrm{d}z \overset{(3)}{=} \pi i + \pi i = 2\pi i$$

(c) The partial fractions ansatz for this function is,

$$f(z) = \frac{C_1}{z} + \frac{C_2}{z+\frac{1}{2}} + \frac{C_3}{z-2},$$

resulting in the linear system,

$$\left.\begin{array}{c} C_1 + C_2 + C_2 = 0 \\ -\frac{3}{2}C_1 - 2C_2 + \frac{1}{2}C_3 = 0 \\ -C_1 = 1, \end{array}\right\} \Longrightarrow (C_1, C_2, C_3) = \left(-1, \frac{4}{5}, \frac{1}{5}\right)$$

Therefore,

$$\oint_C f(z)\,\mathrm{d}z = -1\oint_C \frac{1}{z}\,\mathrm{d}z + \frac{4}{5}\oint_C \frac{1}{z+1/2}\,\mathrm{d}z + \frac{1}{5}\oint_C \frac{1}{z-2}\,\mathrm{d}z \overset{(3)}{=} -2\pi i + \frac{8}{5}\pi i = -\frac{2}{5}\pi i.$$

2.5.3. Evaluate the following integral,

$$\oint_C \frac{e^{iz}}{z(z-\pi)}\,\mathrm{d}z,$$

for each of the following four cases (all circle are centered at the origin; use Eq. (1.2.19) as necessary).

(a) C is the boundary of the annulus between circles of radius 1 and radius 3.
(b) C is the boundary of the annulus between circles of radius 1 and radius 4.
(c) C is a circle of radius R, where $R > \pi$.
(d) C is a circle of radius R, where $R < \pi$.

Solution: Before beginning the exercises, we make the following computation, which will be useful:

$$\begin{aligned} f(z) := \frac{e^{iz}}{z(z-\pi)} &\overset{\text{partial fractions}}{=} -\frac{1}{\pi}\frac{e^{iz}}{z} + \frac{1}{\pi}\frac{e^{iz}}{z-\pi} \\ &= -\frac{1}{\pi}\frac{e^{iz}}{z} + \frac{e^{i\pi}}{\pi}\frac{e^{i(z-\pi)}}{z-\pi} \\ &\overset{\text{Eqn. (1.2.19)}}{=} -\frac{1}{\pi}\frac{\sum_{k=0}^{\infty}\frac{z^k}{k!}}{z} - \frac{1}{\pi}\frac{\sum_{k=0}^{\infty}\frac{(z-\pi)^k}{k!}}{(z-\pi)} \\ &= -\frac{1}{\pi z} - \frac{1}{\pi(z-\pi)} + \underbrace{\frac{1}{\pi}\left(-\sum_{k=1}^{\infty}\frac{z^{k-1}}{k!} - \sum_{k=1}^{\infty}\frac{(z-\pi)^{k-1}}{k!}\right)}_{g(z)} \\ &= -\frac{1}{\pi z} - \frac{1}{\pi(z-\pi)} + g(z), \end{aligned}$$

where $g(z)$ is entire. Since $g(z)$ is entire, then for a curve C enclosing a connected or multiply connected region, we have,

$$\oint_C f(z)\,\mathrm{d}z = -\frac{1}{\pi}\oint_C \left(\frac{1}{z} + \frac{1}{z-\pi}\right)\mathrm{d}z + \oint_C g(z)\,\mathrm{d}z = -\frac{1}{\pi}\oint_C \left(\frac{1}{z} + \frac{1}{z-\pi}\right)\mathrm{d}z. \qquad (4)$$

We will use the above property to compute solutions to this problem:

(a) This curve C does not enclose the points $z = 0$ or $z = \pi$, and hence f is analytic inside the enclosed region, so that by the Cauchy-Goursat Theorem,

$$\oint_C f(z)\,\mathrm{d}z = 0.$$

(b) This curve C encloses the point $z = \pi$, but not $z = 0$. Hence,

$$\begin{aligned}\oint_C f(z)\,\mathrm{d}z &\overset{(4)}{=} -\frac{1}{\pi}\oint_C \frac{1}{z}\,\mathrm{d}z - \frac{1}{\pi}\oint_C \frac{1}{z-\pi}\,\mathrm{d}z \\ &\overset{(3)}{=} -\frac{1}{\pi}(0 + 2\pi i) = -2i.\end{aligned}$$

(c) This region includes both points $z = 0$ and $z = \pi$. Therefore,

$$\begin{aligned}\oint_C f(z)\,\mathrm{d}z &\overset{(4)}{=} -\frac{1}{\pi}\oint_C \frac{1}{z}\,\mathrm{d}z - \frac{1}{\pi}\oint_C \frac{1}{z-\pi}\,\mathrm{d}z \\ &\overset{(3)}{=} -\frac{1}{\pi}(2\pi i + 2\pi i) = -4i.\end{aligned}$$

(d) This region includes $z = 0$, but not $z = \pi$. Therefore,

$$\begin{aligned}\oint_C f(z)\,\mathrm{d}z &\overset{(4)}{=} -\frac{1}{\pi}\oint_C \frac{1}{z}\,\mathrm{d}z - \frac{1}{\pi}\oint_C \frac{1}{z-\pi}\,\mathrm{d}z \\ &\overset{(3)}{=} -\frac{1}{\pi}(2\pi i + 0) = -2i.\end{aligned}$$

2.6.2. Evaluate the integrals $\oint_C f(z)\,\mathrm{d}z$ over a contour C, where C is the boundary of a square with diagonal opposite corners at $z = -(1+i)R$ and $z = (1+i)R$, where $R > a > 0$, and where $f(z)$ is given by the following (use Eq. (1.2.19) in the text as necessary):

(a) $\frac{e^z}{z - \frac{\pi i}{4}a}$

(b) $\frac{e^z}{\left(z - \frac{\pi i}{4}a\right)^2}$

(c) $\frac{z^2}{2z+a}$

(d) $\frac{\sin z}{z^2}$

(e) $\frac{\cosh z}{z}$

Solution: Our main tool in this exercise is the Cauchy integral formula (CIF).

(a) Let $f(z) = e^z$, which is entire. The point $\frac{\pi i}{4}a$ lies inside the square C, so the CIF states,

$$\oint_C \frac{e^z}{z - \frac{\pi i}{4}a}\,\mathrm{d}z = \oint_C \frac{f(z)}{z - \frac{\pi i}{4}a}\,\mathrm{d}z = 2\pi i f\left(\frac{\pi i}{4}a\right) = 2\pi i e^{a i\pi/4}.$$

(b) With the same $f(z) = e^z$ as above, the CIF for derivatives of f implies,

$$\oint_C \frac{e^z}{\left(z - \frac{\pi i}{4}a\right)^2}\,\mathrm{d}z = \oint_C \frac{f(z)}{\left(z - \frac{\pi i}{4}a\right)^2}\,\mathrm{d}z = 2\pi i f'\left(\frac{\pi i}{4}a\right) = 2\pi i e^{a i\pi/4}.$$

(c) We define $f(z) = \frac{z^2}{2}$, which is entire, so that,

$$\oint_C \frac{z^2}{2z+a}\,\mathrm{d}z = \oint_C \frac{f(z)}{z + a/2}\,\mathrm{d}z \overset{\text{CIF}}{=} 2\pi i f(-a/2) = \frac{a^2\pi i}{4},$$

where we have also used the fact that $-a/2$ lies inside C.

(d) Defining $f(z) = \sin z$, which is entire, then

$$\oint_C \frac{\sin z}{z^2}\,\mathrm{d}z = \oint_C \frac{f(z)}{z^2}\,\mathrm{d}z \overset{\text{CIF}}{=} 2\pi i f'(0) = 2\pi i$$

(e) Defining $f(z) = \cosh z$, which is entire, we have,

$$\oint_C \frac{\cosh z}{z}\,\mathrm{d}z = \oint_C \frac{f(z)}{z}\,\mathrm{d}z \overset{\text{CIF}}{=} 2\pi i f(0) = 2\pi i$$

2.6.5. Consider two entire functions with no zeros and having a ratio equal to unity at infinity. Use Liouville's Theorem to show that they are in fact the same function.

Solution: Let f_1 and f_2 be the functions in question, and define,

$$g(z) = \frac{f_1(z)}{f_2(z)},$$

which itself is entire since f_2 has no zeros. Since $\lim_{z\to\infty} g(z) = 1$, then there is some $R \geq 0$ such that

$$|z| > R \implies |g(z) - 1| < \frac{1}{2},$$

which in particular means that,

$$|z| > R \implies |g(z)| < \frac{3}{2}. \tag{5a}$$

Now on $\overline{B_R(0)}$ (the closed origin-centered ball of radius R), g is analytic, and in particular continuous over this closed and bounded set, so that

$$M := \max_{z\in\overline{B_R(0)}} |g(z)| < \infty, \tag{5b}$$

i.e., g is bounded on $\overline{B_R(0)}$. Combining (5a) and (5b) implies,

$$\max_{z\in\mathbb{C}} |g(z)| \leq \max\left\{\frac{3}{2}, M\right\} < \infty,$$

i.e., g is bounded on $\mathbb{C}$ and is analytic on $\mathbb{C}$. By Liouville's theorem, $g(z)$ is constant, and in particular $\lim_{z\to\infty} g(z) = 1$ implies that $g(z) = 1$ over $\mathbb{C}$, i.e., $f_1(z) = f_2(z)$ over $\mathbb{C}$.

2.6.7. Let $f(z)$ be an entire function, with $|f(z)| \leq C|z|$ for all z, where C is a constant. Show that $f(z) = Az$, where A is a constant.

Solution: Our main goal will be to show that $f'' \equiv 0$. Fix an arbitrary $z \in \mathbb{C}$, and let C be a z-centered circle of radius $R > |z|$. The Cauchy Integral formula for the second derivative of f reads,

$$f''(z) = \frac{2!}{2\pi i}\oint_C \frac{f(w)}{(w-z)^3}\,\mathrm{d}w.$$

Using $|f(w)| \leq C|w|$ and parameterizing the integral over the circle with $w(\theta) = z + Re^{i\theta}$, we have,

$$
\begin{aligned}
|f''(z)| &\leq \frac{1}{\pi i} \oint_C \frac{|f(w)|}{|w-z|^3} |\,\mathrm{d}w| \\
&\leq \frac{C}{\pi i} \oint_C \frac{|w|}{|w-z|^3} |\,\mathrm{d}w| \\
&= \frac{C}{\pi i} \int_0^{2\pi} \frac{|z + Re^{i\theta}|}{R^3} R\,\mathrm{d}\theta \\
&\stackrel{|z|<R}{\leq} \frac{C}{\pi R^2} \int_0^{2\pi} 2R\,\mathrm{d}\theta = \frac{4C}{R}
\end{aligned}
$$

This bound is true for every $R > |z|$, i.e., $|f''(z)| < \epsilon$ for every $\epsilon > 0$, implying that $|f''(z)| = f''(z) = 0$. Since z was arbitrary, we have $f''(z) = 0$ for all $z \in \mathbb{C}$. Therefore, $f(z) = Az + B$ for some constants A and B.
The constant B must be zero since $|f(z)| \leq C|z|$ implies that $|f(0)| \leq 0$, i.e., $|B| \leq 0$, so $B = 0$. Hence, $f(z) = Az$.

3.2.3. Let the Euler number E_n be defined by the power series,

$$
\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n.
$$

(a) Find the radius of convergence of this series.
(b) Determine the first six Euler numbers.

Solution:

(a) Since 1 and $\cosh z$ are both entire functions, then $1/\cosh z$ fails to be analytic only where $\cosh z = 0$. The roots of this function correspond to the roots of the equation,

$$
e^z + e^{-z} = 0 \quad \xrightarrow{e^z \neq 0} \quad e^{2z} + 1 = 0.
$$

i.e., $e^{2z} = -1$. Writing this in terms of logarithms, we have,

$$
z = \frac{1}{2} \log -1 = \frac{1}{2}\left(-i\pi + i2\pi k\right) = -\frac{i\pi}{2} + ik\pi,
$$

for every $k \in \mathbb{Z}$. In particular, the two roots that are closest to the origin are,

$$
z = \pm \frac{i\pi}{2}.
$$

In other words, $1/\cosh z$ is analytic on $|z| < R_0$ for every $R_0 < \pi/2$. Hence, the radius of convergence for this power/Taylor series is $R = \pi/2$.

(b) Within the region of convergence of the series, we rewrite it as,

$$
\begin{aligned}
1 &= \cosh z \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n \\
&= \left(\sum_{k=0}^{\infty} c_k z^k\right) \left(\sum_{n=0}^{\infty} \frac{E_n}{n!} z^n\right) \\
&= \sum_{n=0}^{\infty} d_n z^n,
\end{aligned}
$$

where

$$c_k = \begin{cases} \frac{1}{k!}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases} \qquad\qquad d_n = \sum_{k=0}^{n} \frac{E_k}{k!} c_{n-k}.$$

Note that since this must be the power series for the function 1, then

$$d_n = \begin{cases} 1, & n = 0 \\ 0, & n \geq 1. \end{cases}$$

Hence, we can determine the first 6 Euler numbers by equating the two previous expressions for d_n:

$$\begin{aligned}
1 = d_0 &= \frac{E_0}{0!}c_0 = E_0, \\
0 = d_1 &= \frac{E_0}{0!}c_1 + \frac{E_1}{1!}c_0 = E_1 \\
0 = d_2 &= \frac{E_0}{0!}c_2 + \frac{E_1}{1!}c_1 + \frac{E_2}{2!}c_0 = \frac{E_0}{2} + \frac{E_2}{2} \\
0 = d_3 &= \frac{E_0}{0!}c_3 + \frac{E_1}{1!}c_2 + \frac{E_2}{2!}c_1 + \frac{E_3}{3!}c_0 = \frac{E_1}{2} + \frac{E_3}{6} \\
0 = d_4 &= \frac{E_0}{0!}c_4 + \frac{E_1}{1!}c_3 + \frac{E_2}{2!}c_2 + \frac{E_3}{3!}c_1 + \frac{E_4}{4!}c_0 = \frac{E_0}{24} + \frac{E_2}{4} + \frac{E_4}{24} \\
0 = d_5 &= \frac{E_0}{0!}c_5 + \frac{E_1}{1!}c_4 + \frac{E_2}{2!}c_3 + \frac{E_3}{3!}c_2 + \frac{E_4}{4!}c_1 + \frac{E_5}{5!}c_0 = \frac{E_1}{24} + \frac{E_3}{12} + \frac{E_5}{120}
\end{aligned}$$

This is a lower triangular linear system for the unknowns $(E_0, E_1, E_2, E_3, E_4, E_5)$, whose solution is:

$$\begin{pmatrix} E_0 \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 5 \\ 0 \end{pmatrix}$$

Supplement 3.1. Prove the Cauchy Integral Formula: Let f be analytic in an open domain D, and let $z \in D$. Then for any non-negative integer n,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}}\,\mathrm{d}w$$

where C is any simple contour in D enclosing z. You may take the $n = 0, 1$ versions of this formula, proven in class, as given.

Solution: We will proceed by induction over n. The initial case, $n = 0$, is given, so we must only establish the inductive step. Suppose the formula holds for some n at $z \in D$. Then we have for some $\eta \in D$ in a neighborhood of z,

$$\begin{aligned}
\frac{f^{(n)}(\eta) - f^{(n)}(z)}{\eta - z} &= \frac{n!}{2\pi i(\eta - z)} \oint_C f(w) \left[\frac{1}{(w-\eta)^{n+1}} - \frac{1}{(w-z)^{n+1}} \right] \mathrm{d}w \\
&= \frac{n!}{2\pi i(\eta - z)} \oint_C f(w) \left[\frac{(w-z)^{n+1} - (w-\eta)^{n+1}}{(w-z)^{n+1}(w-\eta)^{n+1}} \right] \mathrm{d}w \qquad (6)
\end{aligned}$$

where we take the contour C as, for example, a circular contour enclosing both w and z still contained in D. Using the formula,

$$a^{n+1} - b^{n+1} = (a-b)\left(\sum_{j=0}^{n} a^{n-j} b^j\right),$$

we have,

$$\frac{(w-z)^{n+1} - (w-\eta)^{n+1}}{(w-z)^{n+1}(w-\eta)^{n+1}} = \frac{(\eta - z)\left(\sum_{j=0}^{n}(w-z)^{n-j}(w-\eta)^j\right)}{(w-z)^{n+1}(w-\eta)^{n+1}}$$

Using this in (6) and simplifying, we obtain,

$$\frac{f^{(n)}(\eta) - f^{(n)}(z)}{\eta - z} = \frac{n!}{2\pi i}\oint_C f(w)\left[\frac{\left(\sum_{j=0}^{n}(w-z)^{n-j}(w-\eta)^j\right)}{(w-z)^{n+1}(w-\eta)^{n+1}}\right] \mathrm{d}w$$

Finally, we take the limit as η approaches z (say over any path that remains some bounded distance away from C)):

$$\begin{aligned}
\lim_{\eta\to z}\frac{f^{(n)}(\eta) - f^{(n)}(z)}{\eta - z} &= \frac{n!}{2\pi i}\oint_C f(w)\lim_{\eta\to z}\left[\frac{\left(\sum_{j=0}^{n}(w-z)^{n-j}(w-\eta)^j\right)}{(w-z)^{n+1}(w-\eta)^{n+1}}\right] \mathrm{d}w \\
&= \frac{n!}{2\pi i}\oint_C f(w)\frac{(n+1)(w-z)^n}{(w-z)^{2n+2}}\,\mathrm{d}w \\
&= \frac{(n+1)!}{2\pi i}\oint_C \frac{f(w)}{(w-z)^{n+2}}\,\mathrm{d}w,
\end{aligned}$$

and note that this last integral is well-defined since the integrand is continuous and bounded over the compact contour C. The step of taking the limit under the integral can be formally justified by noting that both $|w-z|$ and $|w-\eta|$ are both bounded below by the (strictly positive) distance from w to C, and hence the integrand is uniformly bounded as $\eta \to z$. Therefore, we may use the bounded/dominated convergence theorem, say over the real and imaginary parts of the integral, to justify taking the limit under the integral. Finally, since this limit by definition is $f^{(n+1)}(z)$, we have completed the inductive step, and the proof.