Department of Mathematics, University of Utah

Applied Complex Variables and Asymptotic Methods

MATH 6720 – Section 001 – Spring 2024

Homework 2 Solutions

Analytic functions and integration, I

Due: Friday, Feb 2, 2024

Below, problem C in section A.B is referred to as exercise A.B.C.
Text: *Complex Variables: Introduction and Applications*, Ablowitz & Fokas,

Exercises: 2.1.1
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2.1.1. Which of the following satisfy the Cauchy-Riemann (C-R) equations? If they satisfy the C-R equations, give the analytic function of z.
(a) $f(x,y) = x - iy + 1$
(b) $f(x,y) = y^3 - 3x^2y + i(x^3 - 3xy^2 + 2)$
(c) $f(x,y) = e^y(\cos x + i\sin y)$

Solution:
(a) With $u(x,y) = x + 1$ and $v(x,y) = -y$, the C-R equations read,

$$\begin{aligned} u_x &= 1 \neq -1 = v_y \\ u_y &= 0 = 0 = -v_x. \end{aligned}$$

Hence, the C-R equations are not satisfied.
(b) With $u(x,y) = y^3 - 3x^2y$ and $v(x,y) = x^3 - 3xy^2 + 2$, the C-R equations read,

$$\begin{aligned} u_x &= -6xy = -6xy = v_y \\ u_y &= 3y^2 - 3x^2 = -3x^2 + 3y^2 = -v_x. \end{aligned}$$

Since u and v satisfy the C-R equations (everywhere), then f is analytic as a function of z. Since

$$iz^3 = i(x+iy)^3 = ix^3 + y^3 - 3x^2y - i3xy^2 = f(x,y) - 2i,$$

then we conclude that $f(x,y) = iz^3 + 2i$.
(c) With $u(x,y) = e^y\cos x$ and $v(x,y) = e^y\sin y$, the C-R equations read,

$$\begin{aligned} u_x &= -e^y\sin x \quad e^y(\sin y + \cos y) = v_y \\ u_y &= e^y\cos x \quad 0 = -v_x. \end{aligned}$$

It is unclear if the two expressions on each line are equal: to determine this, the second set of equations requires

$$e^y \cos x = 0 \implies \cos x = 0 \implies x = \frac{(2k+1)\pi}{2},\ k \in \mathbb{Z}.$$

The first set of equations requires

$$\sin y + \cos y = (-1)^{k+1},$$

Therefore:

$$k \text{ odd} \Longrightarrow \sin y = 1 \text{ or } \cos y = 1 \Longrightarrow y = \frac{(4n+1)\pi}{2}, 2n\pi,\quad n \in \mathbb{Z}.$$
$$k \text{ even} \Longrightarrow \sin y = -1 \text{ or } \cos y = -1 \Longrightarrow y = \frac{(4n+3)\pi}{2}, (2n+1)\pi,\quad n \in \mathbb{Z}.$$

In all of these cases, any (x, y) satisfying the C-R equations is an isolated point, but f is not differentiable in any neighborhood around these points, so f is not analytic.

2.1.5. Let $f(z)$ be analytic in some domain. Show that $f(z)$ is necessarily a constant if either the function $\overline{f(z)}$ is analytic or $f(z)$ assumes only pure imaginary values in the domain.

Solution: Consider first the second case, where we assume both that f is analytic and assumes only purely imaginary values, i.e., $f(z) = iv(x, y)$. By the C-R equations,

$$\begin{aligned} u_x &= 0 = v_y \\ u_y &= 0 = -v_x, \end{aligned}$$

so that $v_x = v_y = 0$, hence $v(x, y) = C$, and so $f(x, y) = iC$, where C must be real since f is purely imaginary-valued.
In the second case, we assume that both $f(z)$ and $\overline{f(z)}$ are analytic. Then with $f(z) = u + iv$ and $\overline{f(z)} = u - iv$, the C-R equations applied to both functions implies,

$$\begin{aligned} u_x &= v_y, & u_y &= -v_x \\ u_x &= -v_y, & u_y &= v_x, \end{aligned}$$

where the first row contains the C-R conditions applied to f, and the second row contains the C-R conditions applied to $\overline{f}$. The first column of equalities implies,

$$v_y = u_x = 0,$$

and the second column implies,

$$v_x = u_y = 0.$$

I.e., $u_x = u_y = 0$ and $v_x = v_y = 0$, so that both u and v must be constant. Thus, $f(z) = u + iv$ is also constant.

2.2.1. Find the location of the branch points and discuss possible branch cuts for the following functions:
(a) $\frac{1}{(z-1)^{1/2}}$

(b) $(z+1-2i)^{1/4}$
(c) $2\log z^2$
(d) $z^{\sqrt{2}}$

Solution:
(a) The branch points of this function are at $z = 1, \infty$. To see why $z = 1$ is a branch point, consider $z = 1 + \epsilon e^{i\theta}$ for a fixed $\epsilon > 0$ and for $\theta \in [0, 2\pi]$. Then:

$$\frac{1}{(z-1)^{1/2}} = (z-1)^{-1/2} = (\epsilon e^{i\theta})^{-1/2} = \frac{1}{\sqrt{\epsilon}} e^{-i\theta/2)}.$$

As θ sweeps from $0 \to 2\pi$, the function sweeps from $\frac{1}{\sqrt{\epsilon}}$ to $\frac{1}{\sqrt{\epsilon}} e^{-i\pi} = -\frac{1}{\sqrt{\epsilon}}$, which is a different value. Hence, $z = 1$ is a branch point. To establish that $z = \infty$ is a branch point, we use the transformation $z - 1 = 1/t$, so the function becomes $\sqrt{t}$, which we already know has a branch point at $t = 0$, i.e., $z = \infty$.

Any simple curve connecting $z = 1$ to $z = \infty$ can serve as a branch cut. E.g., the positive real axis to the right of 1, i.e., $\operatorname{Re}(z) \geq 1$, $\operatorname{Im}(z) = 0$, is a particularly simple choice.

(b) The branch points of this function are $z = -1+2i, \infty$. To establish this, we first consider a mapped version of the function:

$$f(w) = w^{1/4}, \qquad\qquad w = z + 1 - 2i.$$

Note that $f(w)$ has branch points at $w = 0$ and $w = \infty$ (as shown in the text), which correspond to $z = -1 + 2i$ and $z = \infty$, as desired.

Again, any simple curve connecting $z = -1 + 2i$ to $z = \infty$ can seve as a branch cut, and one simple choice can be the semi-infinite ray defined by $\operatorname{Re}(z) \geq -1$, $\operatorname{Im}(z) = 2$.

(c) The branch points of this function are $z = 0, \infty$. To establish this for $z = 0$, take $z = \epsilon e^{i\theta}$ for a small $\epsilon > 0$ and $\theta \in [0, 2\pi]$. At $\theta = 0$ the function takes value $4 \log \epsilon$. As θ sweeps from 0 to 2π, the function takes the value

$$2\log\left(\epsilon e^{i2\pi}\right) = 4\log\epsilon + 4i\pi \neq 4\log\epsilon,$$

showing a discontinuity. Thus, $z = 0$ is a branch point. To establish that $z = \infty$ is a branch point, we make the transformation $w = 1/z$, so that the new function under consideration is

$$f(w) = -2\log(w^2) = 2\log(z^2)$$

By the same arguments as above, $f(w)$ has a branch point at $w = 0$, i.e., the original function has a branch point at $z = \infty$.

Any simple curve connecting $z = 0$ to $z = \infty$ can serve as a branch cut. A simple choice is the positive real axis, $\operatorname{Re}(z) \geq 0$, $\operatorname{Im}(z) = 0$.

(d) The branch points of this function are $z = 0$ and $z = \infty$. Consider the curve $z = \sqrt{\epsilon} e^{i\theta}$ for fixed $\epsilon > 0$ and $\theta \in [0, 2\pi]$. At $\theta = 0$, the function has value $\epsilon^{\sqrt{2}}$. As θ increases from 0 to 2π, the function approaches takes the value,

$$\left(\epsilon e^{i2\pi}\right)^{\sqrt{2}} = \epsilon^{\sqrt{2}} e^{i2\sqrt{2}\pi} \neq \epsilon^{\sqrt{2}},$$

establishing that $z = 0$ is a branch point. To establish that $z = \infty$ is a branch point, use the transformation $z = 1/w$, and consider the function $f(w) = w^{-\sqrt{2}} = z^{\sqrt{2}}$. We can use the same argument as above to show that $f(w)$ has a branch point at $w = 0$, corresponding to $z = \infty$.

Any simple curve connecting $z = 0$ to $z = \infty$ can serve as a branch cut; a simple choice is the positive real axis defined by $\operatorname{Re}(z) \geq 0$, $\operatorname{Im}(z) = 0$.

2.2.2. Determine all possible values and give the principal value of the following numbers (put in the form $x + iy$):
(a) $i^{1/2}$
(b) $\frac{1}{(1+i)^{1/2}}$
(c) $\log(1 + \sqrt{3}i)$
(d) $\log i^3$
(e) $i^{\sqrt{3}}$
(f) $\sin^{-1} \frac{1}{\sqrt{2}}$

Solution:
(a) The function $z \mapsto \sqrt{z}$ has branch points at $0, \infty$ with two branches; we consider any branch cut not passing through i. The possible values, for any $k \in \mathbb{Z}$, are

$$i^{1/2} = \left(e^{i\pi/2 + i2k\pi}\right)^{1/2} = e^{i\pi/4} e^{ik\pi} = \pm e^{i\pi/4} = \pm \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right).$$

The principal value is associated with $k = 0$ above, i.e., the $(+)$ sign choice.
(b) The main branch complexity here stems from the $z \mapsto \sqrt{z}$ map in the denominator, so we compute this first. We again have for any $k \in Z$,

$$(1 + i)^{1/2} = \left(\sqrt{2} e^{i\pi/4 + i2\pi k}\right)^{1/2} = 2^{1/4} e^{i\pi/8 + i\pi k} = \pm 2^{1/4} e^{i\pi/8}$$

Therefore,

$$\frac{1}{(1 + i)^{1/2}} = \pm \frac{e^{-i\pi/8}}{2^{1/4}} = \pm \frac{1}{2^{1/4}} \left(\cos(\pi/8) - i \sin(\pi/8)\right).$$

(c) Since the log function has infinitely many branches, this quantity takes on infinitely many values. For any $k \in \mathbb{Z}$ we have,

$$\log\left(1 + \sqrt{3}i\right) = \log\left(2\left(\frac{1}{2}\frac{\sqrt{3}}{2}i\right)\right) = \log\left(2e^{i\pi/3 + i2\pi k}\right) = \log 2 + i\left(\frac{\pi}{3} + 2\pi k\right).$$

The principal value occurs when $k = 0$.
(d) Again for $k \in \mathbb{Z}$ we directly compute,

$$\log i^3 = \log\left(e^{i3\pi/2 + i2\pi k}\right) = i\left(\frac{3\pi}{2} + 2\pi k\right).$$

The principal value for the log function takes imaginary values on the interval $[-\pi, \pi)$, so the principal value above occurs with $k = -1$, having value $-i\pi/2$.
(e) For any $k \in \mathbb{Z}$ we have,

$$i^{\sqrt{3}} = \left(e^{i\pi/2 + i2\pi k}\right)^{\sqrt{3}} = e^{i\sqrt{3}(\pi/2 + 2\pi k)} = \cos\left(\sqrt{3}\left(\frac{\pi}{2} + 2\pi k\right)\right) + i \sin\left(\sqrt{3}\left(\frac{\pi}{2} + 2\pi k\right)\right).$$

The principal value occurs for $k = 0$.

(f) We express the inverse sine function in terms of the logarithm, and so for any $k \in \mathbb{Z}$ we have,

$$\sin^{-1}\frac{1}{\sqrt{2}} = -i\log\left(\sqrt{1-\frac{1}{2}} + \frac{i}{\sqrt{2}}\right) = -i\log\left(\pm\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \left\{\begin{array}{l} -i\log e^{i\pi/4+2\pi k}, \\ -i\log e^{i3\pi/4+2\pi k}. \end{array}\right.$$

So that we have the following pair of infinite values:

$$\sin^{-1}\frac{1}{\sqrt{2}} = \left\{\begin{array}{l} \frac{\pi}{4}+2\pi k, \\ \frac{3\pi}{4}+2\pi k. \end{array}\right.$$

There are two principal values we must determine: first the principal value of $z \mapsto \sqrt{z}$, and second the principal value of $z \mapsto \log z$. For the first choice, we make the identification $\pm \to +$ in the computations above. Thus, we have,

$$\sin^{-1}\frac{1}{\sqrt{2}} = -i\log e^{i\pi/4+2\pi k}.$$

The principal value of the log function above occurs when $k=0$, so that the principal value is

$$\sin^{-1}\frac{1}{\sqrt{2}} = -i\log e^{i\pi/4} = \frac{\pi}{4}.$$

2.2.3. Solve for z:
(a) $z^5 = 1$
(b) $3 + 2e^{z-i} = 1$
(c) $\tan z = 1$

Solution:
(a) We expect 5 values for z since the function $w \mapsto w^{1/5}$ takes five values. We write $1 = e^{i2\pi k}$ for $k \in \mathbb{Z}$ and then take fifth roots:

$$z^5 = e^{i2\pi k} \implies z = e^{i2\pi k/5} = 1,\ e^{i2\pi/5},\ e^{i4\pi/5},\ e^{i6\pi/5},\ e^{i8\pi/5}.$$

(b) We compute this solution via logarithms. We have for any $k \in Z$:

$$e^{z-i} = -1 \implies z = i + \log(-1) = i + \log e^{i\pi + i2\pi k} = i\left(1 + \pi(2k+1)\right).$$

(c) We use the logarithmic form for the inverse tangent function. As an intermediate step, we compute,

$$\frac{i-1}{i+1} = \frac{\sqrt{2}e^{i3\pi/4}}{\sqrt{2}e^{i\pi/4}} = e^{i\pi/2} = i.$$

Then for any $k \in \mathbb{Z}$ we have

$$z = \tan^{-1} 1 \implies z = \frac{1}{2i}\log\frac{i-1}{i+1} = \frac{1}{2i}\log i = \frac{1}{2i}\log e^{i\pi/2+i2\pi k} = \pi\left(k + 1/4\right).$$

2.4.1. From the basic definition of complex integration, evaluate the integral $\oint_C f(z)\,\mathrm{d}z$, where C is the parameterized unit circle enclosing the origin, $C:\ x(t) = \cos t,\ y(t) = \sin t$ or $z = e^{it}$, and where $f(z)$ is given by,

(a) z^2
(b) $\overline{z}^2$
(c) $\frac{z+1}{z^2}$

Solution:
(a) We parameterize the unit circle with $0 \leq t \leq 2\pi$, and since $z = e^{it}$ use,

$$\mathrm{d}z = ie^{it}\,\mathrm{d}t,$$

to write the integral:

$$\int_C z^2\,\mathrm{d}z = \int_0^{2\pi} \left(e^{it}\right)^2 ie^{it}\,\mathrm{d}t = \int_0^{2\pi} ie^{3it}\,\mathrm{d}t = 0$$

(b) With the same parameterization as the previous part, we have,

$$\int_C \overline{z}^2\,\mathrm{d}z = \int_0^{2\pi} \left(e^{-it}\right)^2 ie^{it}\,\mathrm{d}t = i\int_0^{2\pi} e^{-it}\,\mathrm{d}t = 0$$

(c) With the same parameterization as the previous part, we have,

$$\int_C \frac{z+1}{z^2}\,\mathrm{d}z = \int_0^{2\pi} \frac{e^{it}+1}{e^{2it}} ie^{it}\,\mathrm{d}t = i\int_0^{2\pi} (1+e^{-it})\,\mathrm{d}t = 2\pi i.$$

2.4.4. Use the principal branch of $\log z$ and $z^{1/2}$ to evaluate,
(a) $\int_{-1}^1 \log z\,\mathrm{d}z$
(b) $\int_{-1}^1 z^{1/2}\,\mathrm{d}z$

Solution:
(a) These integrals can be recast as real-valued integrals. To begin, we recall that for a real variable x:

$$\int_0^1 \log x\,\mathrm{d}x = \lim_{\epsilon\downarrow 0}\int_\epsilon^1 \log x\,\mathrm{d}x \stackrel{\text{IbP}}{=} \lim_{\epsilon\downarrow 0}\left(y\log y - y\right)\Big|_\epsilon^1 = \lim_{\epsilon\downarrow 0}\left(-1-\epsilon\log\epsilon+\epsilon\right) = -1.$$

We now write,

$$\int_{-1}^1 \log z\,\mathrm{d}z = \int_{-1}^0 \log z\,\mathrm{d}z + \int_0^1 \log z\,\mathrm{d}z.$$

The second integral, being an integral of a real-valued function over a real interval, takes value -1 as we have already established. Since we are on the principal branch of the logarithm, then $\log z = \log|z| + i\arg z$, where $\arg z \in [-\pi,\pi)$. In our case, $z = |z|e^{-i\pi}$ for $|z| \in [0,1]$, so we have:

$$\int_{-1}^0 \log z\,\mathrm{d}z = \int_{-1}^0 \left(\log|z| - i\pi\right)\,\mathrm{d}z = \int_0^1 \left(\log x - i\pi\right)\,\mathrm{d}x = -i\pi - 1.$$

Putting everything together, we have:

$$\int_{-1}^1 \log z\,\mathrm{d}z = \int_{-1}^0 \log z\,\mathrm{d}z + \int_0^1 \log z\,\mathrm{d}z = -i\pi - 1 - 1 = -2 - i\pi.$$

(b) For the principal branch of the square root function, we treat $z = |z|e^{i \arg z}$, with $\arg z \in [-\pi, \pi)$. I.e., the integral we wish to compute takes the form,

$$\begin{aligned}\int_{-1}^{1} z^{1/2} \, \mathrm{d}z &= \int_{-1}^{0} z^{1/2} \, \mathrm{d}z + \int_{0}^{1} z^{1/2} \, \mathrm{d}z \\ &= e^{-i\pi/2} \int_{-1}^{0} |z|^{1/2} \, \mathrm{d}z + \int_{0}^{1} z^{1/2} \, \mathrm{d}z = (1 - i) \int_{0}^{1} z^{1/2} \, \mathrm{d}z,\end{aligned}$$

where the last equality uses the fact that the value of $|z|^{1/2}$ on $[-1, 0]$ equals (a reflection of) that of $z^{1/2}$ on $[0, 1]$. This last integral is directly computable via the parameterization:

$$z(t) = t, \qquad t \in [0, 1],$$

i.e.,

$$\int_{0}^{1} z^{1/2} \, \mathrm{d}z = \int_{0}^{1} \sqrt{t} \, \mathrm{d}t = \frac{2}{3},$$

and hence,

$$\int_{-1}^{1} z^{1/2} \, \mathrm{d}z = (1 - i)\frac{2}{3}.$$