Department of Mathematics, University of Utah

Applied Complex Variables and Asymptotic Methods

MATH 6720 – Section 001 – Spring 2024

Homework 1 Solutions

Basics of complex numbers

Due Friday, Jan 19, 2024

Submit your solutions online through Gradescope. Below, problem C in section A.B is referred to as exercise A.B.C.

Text: *Complex Variables: Introduction and Applications*, Ablowitz & Fokas,

Exercises:	1.1.2 (b,d)
	1.1.3
	1.2.1
	1.2.5
	Supplement 1.1
	Supplement 1.2

1.1.2. Express each of the following in the form $a + bi$, where a and b are real:

(b) $\frac{1}{1+i}$

(d) $|3 + 4i|$

Solution:

(b) We multiply both numerator and denominator by the complex conjugate:

$$\frac{1}{1+i} = \frac{(1-i)}{(1+i)(1-i)} = \frac{1-i}{2} = \frac{1}{2} - i\frac{1}{2},$$

i.e., $a = 1/2$ and $b = -1/2$.

(d) This is a purely real number:

$$|3 + 4i| = \sqrt{3^2 + 4^2} = 5 = 5 + 0i,$$

so that $a = 5$, $b = 0$.

1.1.3. Solve for the roots of the following equations:

(a) $z^3 = 4$

(b) $z^4 = -1$

(c) $(az + b)^3 = c$, where $a, b, c > 0$

(d) $z^4 + 2z^2 + 2 = 0$

Solution:

(a) We write

$$4 = 4e^{i0} \quad \Longrightarrow \quad z = \sqrt[3]{4}e^{i2\pi/3},\ \sqrt[3]{4}e^{i2\pi/3},\ \sqrt[3]{4}e^{i4\pi/3}$$

(b) We have,

$$-1 = 1e^{i\pi} \quad \Longrightarrow \quad z = e^{i\pi/4},\ e^{i3\pi/4},\ e^{5\pi/4},\ e^{7\pi/4}$$

(c) Letting $w = az + b$, then we seek the roots w such that $w^3 = c$ with $c > 0$. Therefore,

$$w = \sqrt[3]{c},\ \sqrt[3]{c}e^{i2\pi/3},\ \sqrt[3]{c}e^{i4\pi/3}.$$

Therefore, $z = (w - b)/a$ takes values,

$$z = \frac{1}{a}(\sqrt[3]{c} - b),\ \frac{1}{a}(\sqrt[3]{c}e^{i2\pi/3} - b),\ \frac{1}{a}(\sqrt[3]{c}e^{i4\pi/3} - b),$$

(d) We have,

$$z^4 + 2z^2 + 2 = 0 \quad \Longrightarrow \quad (z^2 + 1)^2 + 1 = 0,$$

and hence $z^2 + 1$ are the second roots of -1,

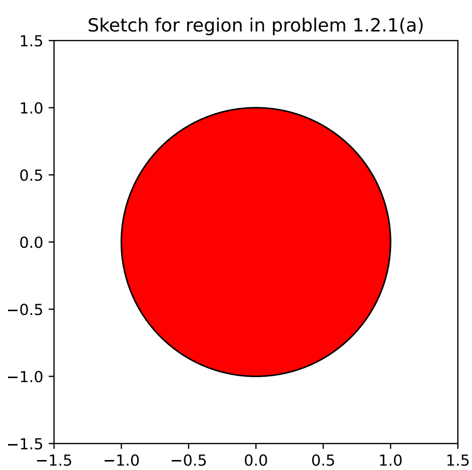
$$z^2 + 1 = e^{i\pi/2},\ e^{i3\pi/2} \quad \Longrightarrow \quad z^2 = -1 \pm i = \sqrt{2}e^{3\pi/4},\ \sqrt{2}e^{5\pi/4}.$$

Thus, z takes on the 4 values,

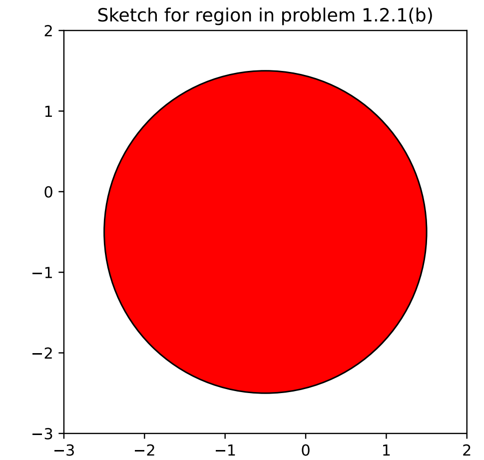
$$z = \sqrt[4]{2}e^{3\pi/8},\ \sqrt[4]{2}e^{11\pi/8}, \qquad z = \sqrt[4]{2}e^{5\pi/8},\ \sqrt[4]{2}e^{13\pi/8}.$$

1.2.1. Sketch the regions associated with the following inequalities. Determine if the region is open, closed, bounded, or compact.
(a) $|z| \leq 1$
(b) $|2z + 1 + i| < 4$
(c) $\mathrm{Re}\,(z) \geq 4$
(d) $|z| \leq |z + 1|$
(e) $0 < |2z - 1| \leq 2$

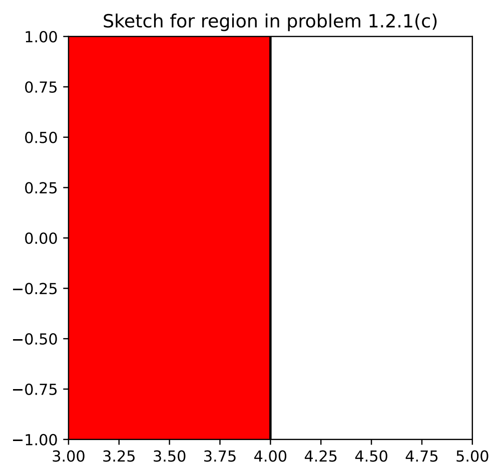
Solution:
(a) The region is closed, bounded, and compact.



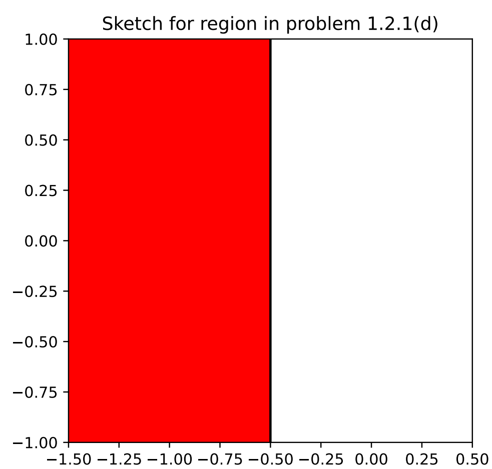
(b) The region is open and bounded.



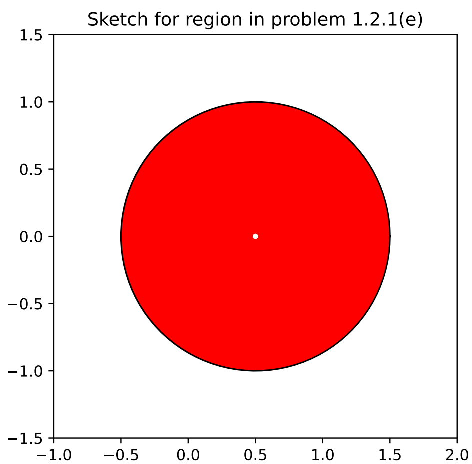
(c) The region is closed.



(d) The region is closed.



(e) The region is bounded.



1.2.5. Use any method to determine series expansions for the following functions:

(a) $\frac{\sin z}{z}$
(b) $\frac{\cosh z - 1}{z^2}$
(c) $\frac{e^z - 1 - z}{z}$

Solution: Using formulas for known power series, we have:

(a)

$$\frac{\sin z}{z} = \frac{1}{z} \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j+1}}{(2j+1)!} = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j}}{(2j+1)!}$$

(b)

$$\frac{\cosh z - 1}{z^2} = \frac{1}{z^2} \left(-1 + \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!} \right) = \frac{1}{z^2} \sum_{j=1}^{\infty} \frac{z^{2j}}{(2j)!} = \sum_{j=1}^{\infty} \frac{z^{2j-2}}{(2j)!}$$

(c)

$$\frac{e^z - 1 - z}{z} = \frac{1}{z} \left(-1 - z + \sum_{j=0}^{\infty} \frac{z^j}{j!} \right) = \frac{1}{z} \sum_{j=2}^{\infty} \frac{z^j}{j!} = \sum_{j=2}^{\infty} \frac{z^{j-1}}{j!} = \sum_{j=0}^{\infty} \frac{z^{j+1}}{(j+2)!}$$

Supplement 1.1. Use de Moivre's Theorem to show that the two expressions,

$$\cos(n\theta), \qquad \frac{\sin((n+1)\theta)}{\sin\theta},$$

for arbitrary $n \in \mathbb{N}$, are both degree-n polynomials of $\cos\theta$.

Solution: We have that,

$$\begin{aligned} \cos(n\theta) = \mathrm{Re}\left(e^{in\theta}\right) = \mathrm{Re}\left(\left(e^{i\theta}\right)^n\right) &= \mathrm{Re}\left((\cos\theta + i\sin\theta)^n\right) \\ &= \mathrm{Re}\left(\sum_{j=0}^{n} i^j \begin{pmatrix} n \\ j \end{pmatrix} \cos^{n-j}\theta \sin^j\theta\right). \end{aligned}$$

Using the fact that i^j is real iff j is even, then by reindexing we have,

$$\begin{aligned}\cos(n\theta) &\overset{k:=\lfloor n/2\rfloor}{=} \sum_{j=0}^{k}(-1)^j \left(\begin{array}{c} n \\ 2j \end{array}\right) \cos^{n-2j}\theta \sin^{2j}\theta \\ &= \sum_{j=0}^{k}(-1)^j \left(\begin{array}{c} n \\ 2j \end{array}\right) \cos^{n-2j}\theta \left(1-\cos^2\theta\right)^j,\end{aligned}$$

and the last expression is clearly a polynomial in $\cos\theta$ of degree $n-2j+2j=n$. The computation for the second expression involving the ratio of sine fucntions is similar:

$$\begin{aligned}\sin((n+1)\theta) &= \mathrm{Im}\left((\cos\theta + i\sin\theta)^{n+1}\right) \\ &= \mathrm{Im}\left(\sum_{j=0}^{n+1} i^j \left(\begin{array}{c} n+1 \\ j \end{array}\right) \cos^{n+1-j}\theta\sin^j\theta\right) \\ &\overset{k:=\lceil (n+1)/2\rceil}{=} \sum_{j=1}^{k}(-1)^{j-1} \left(\begin{array}{c} n+1 \\ 2j-1 \end{array}\right) \cos^{n+1-2j+1}\theta\sin^{2j-1}\theta \\ &= \sin\theta\sum_{j=1}^{k}(-1)^{j-1} \left(\begin{array}{c} n+1 \\ 2j-1 \end{array}\right) \cos^{n-2j+2}\theta\sin^{2j-2}\theta \\ &= \sin\theta\sum_{j=1}^{k}(-1)^{j-1} \left(\begin{array}{c} n+1 \\ 2j-1 \end{array}\right) \cos^{n-2j+2}\theta\,(1-\cos\theta)^{j-1},\end{aligned}$$

Therefore,

$$\frac{\sin((n+1)\theta)}{\sin\theta} = \sum_{j=1}^{k}(-1)^{j-1} \left(\begin{array}{c} n \\ 2j-1 \end{array}\right) \cos^{n-2j+2}\theta\,(1-\cos\theta)^{j-1},$$

which is a polynomial in $\cos\theta$, of degree $n-2j+2+2j-2=n$.

Supplement 1.2. Prove the triangle inequality: Given $z_1, z_2 \in \mathbb{C}$, then

$$||z_1|-|z_2|| \le |z_1+z_2| \le |z_1|+|z_2|.$$

Solution: We compute:

$$\begin{aligned}|z_1+z_2|^2 = (z_1+z_2)\left(\overline{z}_1+\overline{z}_2\right) &= |z_1|^2+|z_2|^2+z_1\overline{z_2}+\overline{z_1}z_2 \\ &= |z_1|^2+|z_2|^2+2\mathrm{Re}\left(z_1\overline{z_2}\right).\end{aligned}$$

We now use the fact that for any $w \in \mathbb{C}$,

$$\mathrm{Re}\,(w) \le |w|.$$

With $w \leftarrow z_1\overline{z_2}$, this implies,

$$-|z_1||z_2| \overset{(A)}{\le} \mathrm{Re}\left(z_1\overline{z_2}\right) \overset{(B)}{\le} |z_1||z_2|.$$

Then on one hand we have,

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}\left(z_1 \overline{z_2}\right) \overset{(A)}{\geq} |z_1|^2 + |z_2|^2 - 2|z_1||z_2| = (|z_1| - |z_2|)^2 = ||z_1| - |z_2||^2,$$

which proves the lower ("reverse") triangle inequality. On the other hand, we have,

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}\left(z_1 \overline{z_2}\right) \overset{(B)}{\leq} |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2,$$

which proves the upper triangle inequality.