Department of Mathematics, University of Utah

Applied Complex Variables and Asymptotic Methods

MTH6720 – Section 01 – Spring 2024

Final exam formula sheet

In what follows, C_R denotes a semicircular arc of radius R in the upper half-plane centered at the origin. The contour C_ϵ is a circular arc of radius ϵ centered around a point z_0 that sweeps out an angle of ϕ.

1. Suppose f is analytic on an open domain containing a simple closed loop C. Then for all integers $n \geq 0$ and all z enclosed by C,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} \, \mathrm{d}w,$$

2. The coefficients for a Laurent series of the function f are given by,

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} \, \mathrm{d}w$$

3. If a continuous f is bounded over a contour C of finite length, i.e., $|f(z)| \leq M < \infty$ for all $z \in C$ and $\int_C |\,\mathrm{d}z| = L < \infty$, then

$$\left| \int_C f(z) \, \mathrm{d}z \right| \leq ML$$

4. Suppose $f(z) = P(z)/Q(z)$ is a rational function with $\deg Q \geq \deg P + 2$. Then,

$$\lim_{R\to\infty} \int_{C_R} f(z) \, \mathrm{d}z = 0.$$

5. (Jordan's Lemma) Suppose that $f(z) \to 0$ uniformly for $z \in C_R$ as $R \to \infty$. Then for any $k > 0$,

$$\lim_{R\to\infty} \int_{C_R} e^{ikz} f(z) \, \mathrm{d}z = 0.$$

6. Suppose that $(z - z_0)f(z) \to 0$ uniformly for $z \in C_\epsilon$ as $\epsilon \to 0$. Then,

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} f(z) \, \mathrm{d}z = 0.$$

7. Suppose that f has a simple pole at $z = z_0$. Then

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} f(z) \, \mathrm{d}z = i\phi \operatorname{Res}(f; z_0).$$

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8. With C_R any origin-centered circular arc (not necessarily in the upper half-plane), if $zf(z) \to 0$ uniformly on C_R as $R \to 0$, then,

$$\lim_{R\to\infty} \int_{C_R} f(z)\,\mathrm{d}z = 0.$$

Laplace-type integrals These are formulas regarding asymptotic ($k \to \infty$) behavior of $I(k) := \int_a^b f(t)e^{-k\phi(t)}\,\mathrm{d}t$ for $a < b$.

a. (Watson's Lemma) Set $a = 0$ and $\phi(t) = t$. Assume f is integrable with the series expansion,

$$f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^{\beta n} \quad t \to 0^+, \quad \alpha > -1, \quad \beta > 0.$$

In addition, if $b < \infty$ then assume $|f(t)| \leq M < \infty$ for $t \in [a, b]$, and if $b = \infty$ then assume $f(t) = \mathcal{O}(e^{ct})$ as $t \to \infty$ for some $c \in \mathbb{R}$. Then,

$$I(k) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha + \beta n + 1)}{k^{\alpha+\beta n+1}}.$$

b. (Laplace's Method) Assume $b < \infty$, and that $\phi \in C^4([a, b])$ and $f \in C^2([a, b])$. Suppose that for some $c \in [a, b]$, we have $\phi'(c) = 0$ and $\phi''(c) > 0$. Also, assume that $\phi'(t) \neq 0$ for all $t \in [a, b]\backslash\{c\}$. Then,

$$I(k) \sim G(c)e^{-k\phi(c)}f(c)\sqrt{\frac{2\pi}{k\phi''(c)}} + \mathcal{O}\left(\frac{e^{-k\phi(c)}}{k^{G(c)+1/2}}\right), \qquad G(c) := \left\{ \begin{array}{ll} 1, & c \in (a, b) \\ \frac{1}{2}, & \text{otherwise} \end{array} \right.$$

Fourier-type integrals These are formulas regarding asymptotic ($k \to \infty$) behavior of $I(k) := \int_a^b f(t)e^{ik\phi(t)}\,\mathrm{d}t$ for $a < b$.

a. Set $a = 0$, and $\phi(t) = \mu t$, where $\mu = \pm 1$, and $k > 0$. Suppose f vanishes infinitely smoothly at $t = b$, that $f \in C^{\infty}((0, b])$, and that for some $\gamma > -1$, $f(t) \sim t^{\gamma} + o(t^{\gamma})$ as $t \to 0^+$. Then,

$$I(k) = \left(\frac{1}{k}\right)^{\gamma+1} \Gamma(\gamma + 1)e^{i\frac{\pi}{2}\mu(\gamma+1)} + o(k^{-(\gamma+1)}).$$

b. (Stationary phase) Suppose $c \in [a, b]$ is the only value of t where $\phi'(t)$ vanishes. Assume that f vanishes infinitely smoothly at both $t = a$ and $t = b$, and that both f and ϕ are C^{∞} on the intervals $[a, c)$ and $(c, b]$. Suppose that there is some $\gamma > -1$ such that as $t \to c$,

$$\begin{aligned} \phi(t) - \phi(c) &\sim \alpha(t - c)^2 + o((t - c)^2), \\ f(t) &\sim \beta(t - c)^{\gamma} + o((t - c)^{\gamma}). \end{aligned}$$

Then with $\mu = \operatorname{sgn} \alpha$,

$$\int_a^b f(t)e^{ik\phi(t)}\,\mathrm{d}t \sim e^{ik\phi(c)}\beta\Gamma\left(\frac{\gamma + 1}{2}\right) e^{i\pi\frac{\gamma+1}{4}\mu} \left(\frac{1}{k|\alpha|}\right)^{\frac{\gamma+1}{2}} + o\left(k^{-\frac{\gamma+1}{2}}\right).$$