Math 6630: Analysis of Numerical Methods, II
Solvers for initial value problems, Part IV

See Ascher and Petzold 1998, Chapters 1-5

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$$\boldsymbol{u}'(t) = \boldsymbol{f}(t;\boldsymbol{u}), \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0.$$

$$\boldsymbol{u}_n \approx \boldsymbol{u}(t_n)$$

$$\boldsymbol{u}_{n+1} \approx \boldsymbol{u}_n + \int_{t_n}^{t_{n+1}} \boldsymbol{f}(t,\boldsymbol{u}(t))\mathrm{d}t$$

We have previously discussed

- Simple schemes: forward/backward Euler, Crank-Nicolson
- Consistency and LTE
- 0-stability and scheme convergence
- absolute/A-stability and consequences
- multi-stage (Runge-Kutta) methods

Finally, we'll discuss multi-step schemes.

Preliminaries: polynomial interpolation

To begin we review some basic concepts about (univariate) polynomial interpolation:

Let $f : \mathbb{R} \to \mathbb{R}$ be a scalar function, and let $x_0, \ldots, x_n$ be any distinct points on $\mathbb{R}$.

Theorem

There is a unique polynomial $p(x)$ of degree n such that $f(x_j) = p(x_j)$ for all $j = 0, \ldots, n$.

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One way to construct this polynomial is via divided differences. Define

$$f[x_j] = f(x_j), \quad f[x_j, \ldots, x_{j+\ell}] = \frac{f[x_{j+1}, \ldots, x_{j+\ell}] - f[x_j, \ldots, x_{j+\ell-1}]}{x_{j+\ell} - x_j},$$

which are approximations to ℓth derivatives. Then,

$$p(x) = \sum_{\ell=0}^{n} f[x_0, \ldots, x_j] \prod_{j=0}^{\ell-1} (x - x_j).$$

This is the Newton form of the interpolating polynomial.

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If $x_j = x_0 + jk$ for some $k > 0$, then expressions simplify considerably and more explicit formulas can be derived.

Simple theory for linear difference equations parallels linear differential equations:

$$u^{(s)}(t) + \sum_{j=1}^{s} \alpha_j u^{(s-j)}(t) = 0, \qquad u^{(j)}(0) = u_0^j, \qquad j = 0, \ldots, s-1.$$

Solve for a function $u(t)$, $t > 0$. The order is $s > 0$.

$$u_n + \sum_{j=1}^{s} \alpha_j u_{n-j} = 0, \qquad u_{n-j} = u_{n-j,0}, \qquad j = 1, \ldots, s.$$

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$$\text{Ansatz } u(t) = e^{zt} \implies p(z) := \sum_{j=0}^{s} \alpha_j z^{s-j} = 0, \quad (\alpha_0 = 1)$$

Solutions take the form $u(t) \sim e^{z_j t}$, where $z_1, \ldots, z_s$ are the roots of p.

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Solutions $u(t)$ are stable if $\Re z_j \leqslant 0$. (Asymptotically stable if $\Re z_j < 0$.)

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For the IVP,

$$\boldsymbol{u}'(t) = \boldsymbol{f}(t; \boldsymbol{u}), \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0.$$

$$\boldsymbol{u}_n \approx \boldsymbol{u}(t_n)$$

a general s-step multi-step scheme with timestep k has the form,

$$\frac{du}{dt} \simeq \sum_{j=0}^{s} \alpha_j \boldsymbol{u}_{n+1-j} = k \sum_{j=0}^{s} \beta_j \boldsymbol{f}(t_{n+1-j}, \boldsymbol{u}_{n+1-j}), \simeq \int_{t_n}^{t_{n+1}} f(t, u(t))\, dt \qquad \alpha_j, \beta_j \in \mathbb{R}$$

Multi-step methods, I

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Comments:

- $s = 1$ corresponds to a general single-step (and single-stage) method

$$\alpha_0 u_{n+1} + \alpha_1 u_n = k\beta_0 f(t_{n+1}, u_{n+1}) + k\beta_1 f(t_n, u_n)$$

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- We assume $\alpha_0 \neq 0$. $\alpha_0 u_{n+1}$
- We can rescale the equation by a constant without changing anything: we fix this freedom by setting $\alpha_0 = 1$.

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- To avoid some minor pathologies, we typically assume that either $\alpha_j \neq 0$ or $\beta_j \neq 0$ for every j.
- $\beta_0 \neq 0$ corresponds to an implicit method. $\beta_0 = 0$ is an explicit method.

Multi-step methods, II

To simplify notation, we will assume the ODE is autonomous ($\boldsymbol{f}(t, \boldsymbol{u}) = \boldsymbol{f}(\boldsymbol{u})$), and will abbreviate $\boldsymbol{f}(\boldsymbol{u}_j)$ as $\boldsymbol{f}_j$. Then the multi-step method takes the form,

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Generally speaking, the constants are chosen so that:

- The α_j approximate $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t_n)$
- The β_j approximate $\frac{1}{k}\int_{t_n}^{t_{n+1}} \boldsymbol{f}(\boldsymbol{u}(r))\mathrm{d}r$

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There are some miscellaneous issues we'll answer later, e.g.,

- If $s \geqslant 2$, how is $\boldsymbol{u}_1$ computed from $\boldsymbol{u}_0$?
- Must we fix the time-step k?

A warmup: single-step specializations

Specializing to single-step methods ($s = 1$) yields a transparent family of methods:

$$\boldsymbol{u}_{n+1} + \alpha_1 \boldsymbol{u}_n = k \left(\beta_0 \boldsymbol{f}_{n+1} + \beta_1 \boldsymbol{f}_n \right).$$

(Recall $\alpha_0 = 1$)

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With this restriction, then we have

$$\boldsymbol{u}_{n+1} - \boldsymbol{u}_n = k \left(\beta_0 \boldsymbol{f}_{n+1} + \beta_1 \boldsymbol{f}_n \right),$$

and hence the right hand side should approximate $\int_{t_n}^{t_{n+1}} \boldsymbol{f}(\boldsymbol{u}(r)) \mathrm{d}r$, requiring $\beta_0 + \beta_1 = 1$ for consistency.

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Then our general family of methods is

$$\boldsymbol{u}_{n+1} = \boldsymbol{u}_n + k \left(\beta \boldsymbol{f}_{n+1} + (1 - \beta) \boldsymbol{f}_n \right),$$

specializing to,

- $\beta = 0$: Forward Euler
- $\beta = 1$: Backward Euler
- $\beta = 1/2$: Crank-Nicolson

Can we derive 2-step methods?

$s = 2$

$$\alpha_0 U_{n+1} + \alpha_1 U_n + \alpha_2 U_{n-1} = k\beta_0 f_{n+1} + k\beta_1 f_n + k\beta_2 f_{n-1}$$

- set $\alpha_0 = 1$
- explicit scheme: $\beta_0 = 0$.

$$U_{n+1} + \alpha_1 U_n + \alpha_2 U_{n-1} = k\beta_1 f_n + k\beta_2 f_{n-1} \qquad \alpha_1, \alpha_2, \beta_1, \beta_2 = ?$$

enforce consistency by "minimizing" LTE.

$$\text{LTE: } \frac{1}{k}\left(u(t_{n+1}) + \alpha_1 u(t_n) + \alpha_2 u(t_{n-1})\right) - \beta_1 f(t_n, u(t_n)) - \beta_2 f(t_{n-1}, u(t_{n-1}))$$

Taylor expand @ $t = t_{n-1}$

abbreviations: $u = u(t_{n-1})$
$u' = u'(t_{n-1})$
$\vdots$
$u^{(j)} = u^{(j)}(t_{n-1})$

$$u(t_{n+1}) = u + 2ku' + \frac{(2k)^2}{2} u'' + \frac{(2k)^3}{6} u''' + \dots$$

using: $u' = f$, $u^{(j)} = \frac{d^{j-1}}{dt^{j-1}} f(t, u(t))$

$$f(t_n, u(t_n)) = f + kf' + k^2/2\, f'' + k^3/6\, f''' + \dots$$
$$= u' + ku'' + \frac{k^2}{2} u''' + \dots$$

$$\text{LTE: } \frac{1}{k}\left(u + 2ku' + 2k^2 u'' + \frac{4}{3}k^3 u''' + \dots\right)$$
$$+ \frac{\alpha_1}{k}\left(u + ku' + k^2/2\, u'' + k^3/6\, u''' + \dots\right)$$
$$+ \frac{\alpha_2}{k} u$$

$$-\beta_1 \left[u' + k u'' + \frac{k^2}{2} u''' + \dots \right]$$

$$-\beta_2 u'$$

$\frac{1}{k}$: $1 + \alpha_1 + \alpha_2 = 0$

1: $2 + \alpha_1 - \beta_1 - \beta_2 = 0$

k: $2 + \alpha_1/2 - \beta_1 = 0 \longrightarrow 4 + \alpha_1 - 2\beta_1 = 0$

k^2: $\frac{4}{3} + \frac{\alpha_1}{6} - \beta_1/2 = 0 \longrightarrow 8 + \alpha_1 - 3\beta_1 = 0$

$$\Downarrow$$

$$4 - \beta_1 = 0$$

$$\beta_1 = 4, \quad \alpha_1 = 4$$

$$\alpha_2 = -5, \quad \beta_2 = 2$$

$$\Rightarrow U_{n+1} + 4U_n - 5U_{n-1} = 4f_n + 2f_{n-1}$$

$$\text{LTE} = O(k^3)$$

(This didn't work...??)

The Adams Family

There are two major classes of most popular multi-step methods. The first is the family of *Adams* methods.

For these methods we start with,

$$\boldsymbol{u}(t_{n+1}) = \boldsymbol{u}(t_n) + \int_{t_n}^{t_{n+1}} \boldsymbol{f}(\boldsymbol{u}(r))\mathrm{d}r,$$

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The β_j are chosen as a quadrature rule to approximate the integral:

$$\int_{t_n}^{t_{n+1}} \boldsymbol{f}(\boldsymbol{u}(r))\mathrm{d}r \approx k \sum_{j=0}^{s} \beta_j \boldsymbol{f}_{n+1-j}$$

Note that we are using points *outside* the interval of intergration (if $s > 1$).

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Note that we are using points *outside* the interval of intergration (if $s > 1$). Again, the particular type of scheme depends on whether we want an implicit or an explicit method:

- $\beta_0 = 0$ yields explicit methods (one fewer parameter to invest in LTE reduction)
- $\beta_0 \neq 0$ yields implicit methods

Adams-Bashforth Methods

The choice of explicit path yields the family of Adams-Bashforth methods.

$$\boldsymbol{u}_{n+1} - \boldsymbol{u}_n = k \sum_{j=1}^{s} \beta_j \boldsymbol{f}_{n+1-j}.$$

The β_j coefficients are used to ensure high-order LTE. E.g., two equivalent strategies:

- Expand in Taylor series, match terms by setting β_j
- Interpolate a degree-$(s-1)$ polynomial on data at $t_{n+1-s}, \ldots, t_n$, integrate the polynomial. The resulting coefficients multiplying the data are the β_j.

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Coefficients for the Adams-Bashforth methods with order=steps:

	β_1	β_2	β_3	β_4	β_5	β_6
$p = s = 1$	1					
$p = s = 2$	$\frac{3}{2}$	-1				
$p = s = 3$	$\frac{23}{12}$	$-\frac{16}{12}$	$\frac{5}{12}$			
$p = s = 4$	$\frac{55}{24}$	$-\frac{59}{24}$	$\frac{37}{24}$	$-\frac{9}{24}$		
$p = s = 5$	$\frac{1901}{720}$	$-\frac{2774}{720}$	$\frac{2616}{720}$	$-\frac{1274}{720}$	$\frac{251}{720}$	
$p = s = 6$	$\frac{4277}{1440}$	$-\frac{7923}{1440}$	$\frac{9982}{1440}$	$-\frac{7298}{1440}$	$\frac{2877}{1440}$	$-\frac{475}{1440}$

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Note that technically we can take $s = 0$ here, which yields backward Euler. (Though you'd still call this a 1-step method.) Coefficients for the Adams-Moulton methods with order=steps+1:

	β_0	β_1	β_2	β_3	β_4	β_5
$p-1=s=1$	$\frac{1}{2}$	$\frac{1}{2}$				
$p-1=s=2$	$\frac{5}{12}$	$\frac{8}{12}$	$-\frac{1}{12}$			
$p-1=s=3$	$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$		
$p-1=s=4$	$\frac{251}{720}$	$\frac{646}{720}$	$-\frac{264}{720}$	$\frac{106}{720}$	$-\frac{19}{720}$	
$p-1=s=5$	$\frac{475}{1440}$	$\frac{1427}{1440}$	$-\frac{798}{1440}$	$\frac{482}{1440}$	$-\frac{173}{1440}$	$\frac{27}{1440}$

Backward Differentiation formulas

The Adams family of methods is not particularly robust for stiff problems.

As an alternative, consider the general form:

$$\sum_{j=0}^{s} \alpha_j \boldsymbol{u}_{n+1-j} = k \sum_{j=0}^{s} \beta_j \boldsymbol{f}(t_{n+1-j}, \boldsymbol{u}_{n+1-j}),$$

and now instead let us focus effort on setting $\beta_j = 0$ for $j > 0$, and choosing α_j to approximate $y'(t_n)$ to high order:

$$\sum_{j=0}^{s} \alpha_j \boldsymbol{u}_{n+1-j} = k\beta_0 \boldsymbol{f}_{n+1}.$$

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	β_0	α_0	α_1	α_2	α_3	α_4	α_5	α_6
$p = s = 1$	1	1	-1					
$p = s = 2$	$\frac{2}{3}$	1	$-\frac{4}{3}$	$\frac{1}{3}$				
$p = s = 3$	$\frac{6}{11}$	1	$-\frac{18}{11}$	$\frac{9}{11}$	$-\frac{2}{11}$			
$p = s = 4$	$\frac{12}{25}$	1	$-\frac{48}{25}$	$\frac{36}{25}$	$-\frac{16}{25}$	$\frac{3}{25}$		
$p = s = 5$	$\frac{60}{137}$	1	$-\frac{300}{137}$	$\frac{300}{137}$	$-\frac{200}{137}$	$\frac{75}{137}$	$-\frac{12}{137}$	
$p = s = 6$	$\frac{60}{147}$	1	$-\frac{360}{147}$	$\frac{450}{147}$	$-\frac{400}{147}$	$\frac{225}{147}$	$-\frac{72}{147}$	$\frac{10}{147}$

Consistency and order of approximation

It's much easier to compute order conditions for multi-step methods (compared to multi-stage ones).

In particular, to compute the LTE for the scheme,

$$\sum_{j=0}^{s} \alpha_j \boldsymbol{u}_{n+1-j} = k \sum_{j=0}^{s} \beta_j \boldsymbol{f}(t_{n+1-j}, \boldsymbol{u}_{n+1-j}),$$

we need to compute the residual for the expression

$$\frac{1}{k} \sum_{j=0}^{s} \alpha_j \boldsymbol{u}(t_{n+1-j}) - \sum_{j=0}^{s} \beta_j \boldsymbol{f}(t_{n+1-j}, \boldsymbol{u}(t_{n+1-j})).$$

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Noting that $\boldsymbol{u}'(t) = \boldsymbol{f}(t, \boldsymbol{u}(t))$, the above expression is equivalent to,

$$\frac{1}{k} \sum_{j=0}^{s} \alpha_j \boldsymbol{u}(t_{n+1-j}) - \sum_{j=0}^{s} \beta_j \boldsymbol{u}'(t_{n+1-j}),$$

and hence we can compute order conditions simply by computing Taylor expansions of $\boldsymbol{u}$ and $\boldsymbol{u}'$.

$$\frac{1}{k}\sum_{j=0}^{s}\alpha_j \boldsymbol{u}(t_{n+1-j}) - \sum_{j=0}^{s}\beta_j \boldsymbol{u}'(t_{n+1-j}),$$

The $\mathcal{O}(1/k)$ terms from the above come from Taylor expansions of the α_j terms, implying that we require,

$$\sum_{j=0}^{s}\alpha_j = 0.$$

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For consistency (LTE vanishing as $k \downarrow 0$), we likewise require the $\mathcal{O}(1)$ terms to vanish, i.e.,

$$\sum_{j=0}^{s}(s-j)\alpha_j - \sum_{j=0}^{s}\beta_j = 0.$$

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These two expressions are evaluations of certain *characteristic* polynomials:

$$\left.\begin{array}{l}\rho(w) = \sum_{j=0}^{s}\alpha_j w^{s-j} \\ \sigma(w) = \sum_{j=0}^{s}\beta_j w^{s-j}\end{array}\right\} \Longrightarrow \begin{array}{l}\rho(1) = 0 \\ \rho'(1) = \sigma(1)\end{array}$$

Consistency of multi-step methods, II

$$\mathrm{LTE} = \frac{1}{k}\sum_{j=0}^{s} \alpha_j \boldsymbol{u}(t_{n+1-j}) - \sum_{j=0}^{s} \beta_j \boldsymbol{u}'(t_{n+1-j}),$$

$$\rho(w) = \sum_{j=0}^{s} \alpha_j w^{s-j}$$

$$\sigma(w) = \sum_{j=0}^{s} \beta w^{s-j}$$

We have shown the following:

Theorem

A multi-step method is consistent if and only if $\rho(1) = 0$ *and* $\rho'(1) = \sigma(1)$.

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We have shown the following:

Theorem

A multi-step method is consistent if and only if $\rho(1) = 0$ and $\rho'(1) = \sigma(1)$.

Of course, to attain more than first-order accuracy, we require more conditions.

0-Stability of multi-step methods, I

The characteristic polynomials are also integral in determining 0-stability:

Theorem

An s-step linear multi-step method is 0-stable if and only if the roots $w_1, \ldots, w_s$ of $\rho(w)$ all satisfy $|w_i| \leqslant 1$, and any roots satisfying $|w_i| = 1$ are simple. (Terminologically: "ρ satisfies the root condition")

This gives a fairly computable condition to identify 0-stability.

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Why is this related to 0-stability? Recall that the essential message of 0-stability is that

Initial data perturbations of size ϵ lead to numerical solutions with errors $C\epsilon$ for small enough ϵ.

(I.e., the k-asymptotic LTE behavior bounds the actual error in the numerical method up to a constant.)

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The above is actually a more abstract version of the definition of 0-stability compared to what we saw in slides `D06`.

Note: we only require control of perturbations for vanishingly small ϵ.

It turns out that while phrased as perturbations to initial data, this is conceptually similar to perturbations of $\boldsymbol{f}$, and under an ODE well-posedness result, is equivalent to considering perturbations of $\boldsymbol{f} = \boldsymbol{0}$.

I.e., it's enough to check controlled perturbations for $\boldsymbol{u}' = \boldsymbol{0}$
(This is why it's called 0-stability.)

0-Stability of multi-step methods, II

So if we only need to consider a linear multistep method for $\boldsymbol{u}' = \boldsymbol{0}$ to account for 0-stability, this means the scheme reads,

$$\sum_{j=0}^{s} \alpha_j u_{n+1-j} = 0,$$

say with starting conditions $u_0 = \cdots = u_{s-1} = 0$.

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If we replace the initial data of 0's by perturbations, then the exact solution to the difference equation, assuming unique roots $w_1, \ldots, w_s$ of ρ, is,

$$u_n \sim \epsilon_1 w_1^n + \cdots + \epsilon_s w_s^n$$

where $\epsilon_1, \ldots, \epsilon_s$ are dependent on the initial data perturbations.

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The exact solution to this problem is $u_n = 0$. As $k \downarrow 0$, then $n \uparrow \infty$. I.e., our perturbed solution is bounded relative to 0 iff $|w_j| < 1$ for all j.

$|w_j| = 1$ is allowed for simple roots, but for repeated roots with multiplicity m, then $|u_n| \sim n^{m-1}|w_j| = n^{m-1}$, which is unbounded in n for $m > 1$.

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Hence, 0-stability is equivalent to the roots of ρ lying within the unit circle (and on the boundary for simple roots).

0-stability examples

Example: any one-step ($s = 1$) method is 0-stable, since $\rho(w) = w - 1$.

$$u_{n+1} + \alpha_1 u_n = \beta_0 f_{n+1} + \beta_1 f_n$$

$\Downarrow$ $\alpha_1 = -1$ by consistency

$$\rho(w) = w - 1$$

0-stability examples

Example: any one-step ($s = 1$) method is 0-stable, since $\rho(w) = w - 1$.

Example: All Adams- methods are 0-stable, since $\rho(w) = w^s - w^{s-1}$.

$$u_{n+1} + \alpha_1 u_n = k \sum_{j=0}^{s} \beta_j f_{n+1-j}$$

$$\alpha_1 = -1 \text{ (consistency)} \Rightarrow \rho(w) = w^s - w^{s-1} = w^{s-1}(w-1)$$

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Example: All BDF methods for $s \leqslant 6$ are 0-stable. Any BDF method with $s > 6$ is unstable.

There are reasonable-looking methods that violate 0-stability:

$$\boldsymbol{u}_{n+1} + 4\boldsymbol{u}_n - 5\boldsymbol{u}_{n-1} = k\left(4\boldsymbol{f}_n + 2\boldsymbol{f}_{n-1}\right),$$

and these methods are actually quite unstable.

$\rho(w) = w^2 + 4w - 5 = (w+5)(w-1)$

$w = 1, \; -5$

Convergence

Just like our analysis for simple Euler-type schemes, 0-stability and consistency *are* convergence.

Theorem (Dahlquist Equivalence Theorem)

Consider an s-step multistep method where the startup values $\boldsymbol{u}_0, \ldots, \boldsymbol{u}_{s-1}$ are generated in a consistent way ($\boldsymbol{u}_j \to \boldsymbol{u}(0)$ as $k \downarrow 0$ for $j = 0, \ldots, s-1$.)

Such a linear multistep method is convergent if and only if it is consistent and 0-stable.

I.e.,:

A linear multistep method is convergent if and only if $\rho(1) = 0$, $\rho'(1) = \sigma(1)$, and ρ satisfies the root condition.

(When convergent, a linear multistep method has order of convergence equal to the order p of the LTE.)

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Note that 1 is *always* a root of ρ for multistep methods of interest.

Methods for which 1 is the *only* unity-modulus root are *strongly* stable. (Otherwise, they are weakly stable.)

Absolute stability

We have a similar notion of absolute stability for multi-step methods: We require that the iteration,

$$\sum_{j=0}^{s} \alpha_j \boldsymbol{u}_{n+1-j} = k \sum_{j=0}^{s} \beta_j \boldsymbol{f}(t_{n+1-j}, \boldsymbol{u}_{n+1-j}),$$

produces solutions $\boldsymbol{u}_n$ that do not grow exponentially in n for the test equation $u' = \lambda u$.

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This results in the difference equation,

$$\sum_{j=0}^{s} \alpha_j \boldsymbol{u}_{n+1-j} = k\lambda \sum_{j=0}^{s} \beta_j \boldsymbol{u}_{n+1-j},$$

whose characteristic equation is,

$$\rho(w) = k\lambda\sigma(w) \stackrel{z=\lambda k}{=} z\sigma(w).$$

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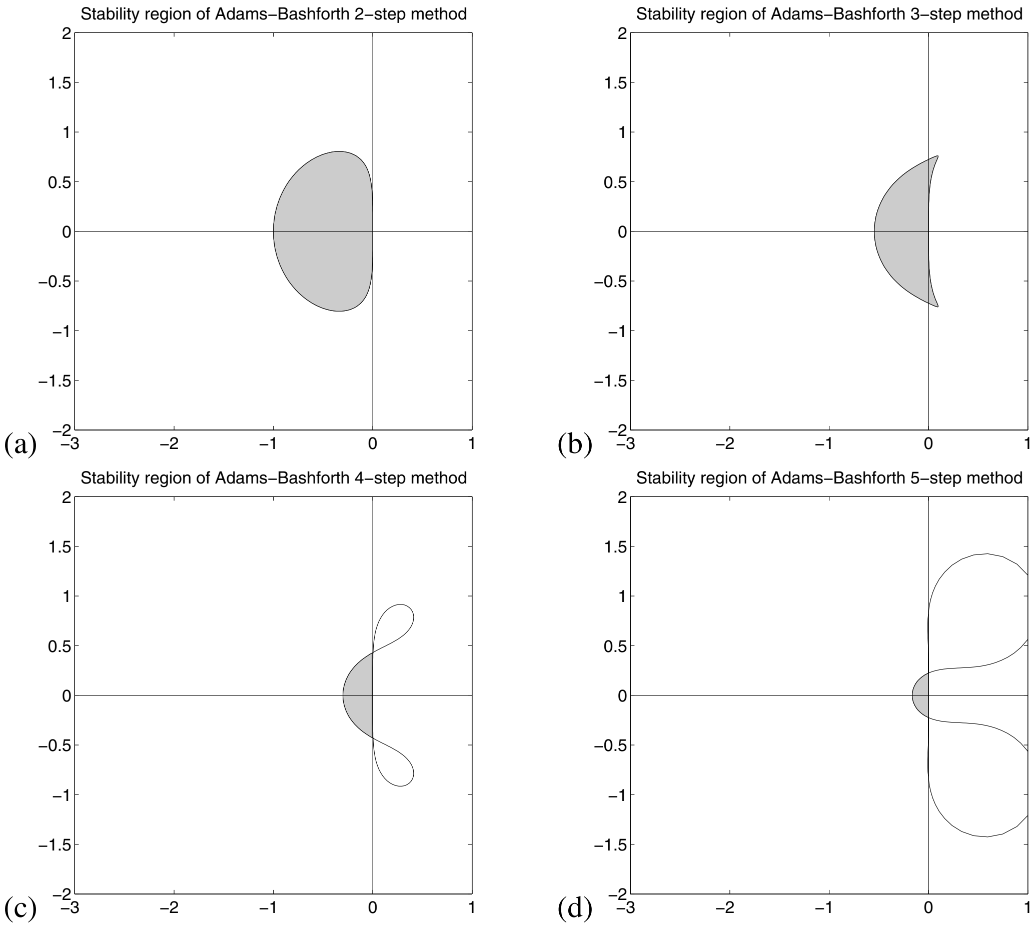
Thus, we say that the region of (absolute) stability for the scheme is the set of z values such that $\rho(w) - z\sigma(w)$ has roots $w_1, \ldots, w_s$ all satisfying $|w_j| < 1$, with $|w_j| = 1$ allowed for simple roots. (I.e., $z \in \mathrm{ROS}$ means $\rho(\cdot) - z\sigma(\cdot)$ satsifies the root condition.)

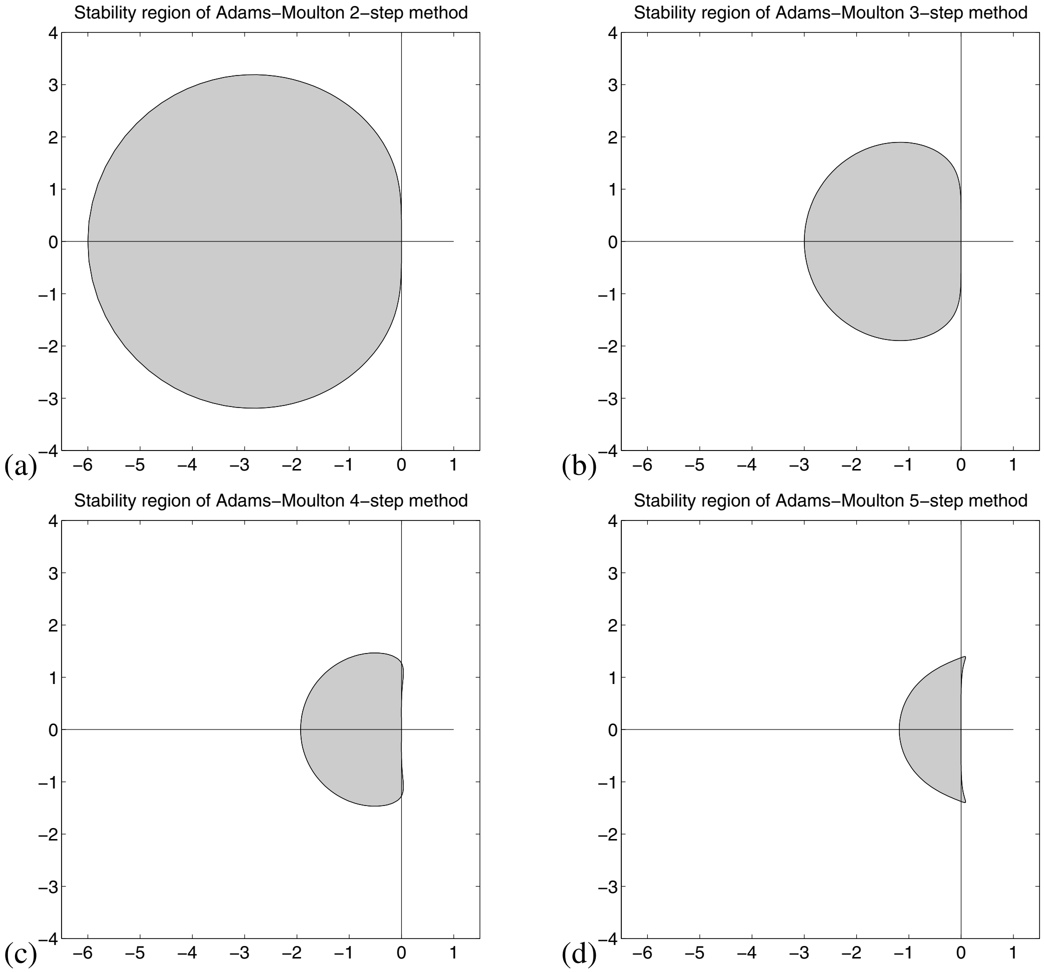
(Note that $0 \in \mathrm{ROS} \iff 0$-stability.)

Absolute stability: Adams-Bashforth



LeVeque 2007, Figure 7.2

Absolute stability: Adams-Moulton



LeVeque 2007, Figure 7.3

Dahlquist barriers

There are some somewhat dissapointing results about efficacy of multistep methods:

Theorem (First Dahlquist Barrier)

For an s-step multistep method that is 0-stable:

- *An explicit method can have order of accuracy at most $p = s$.*
- *If s is odd, the order of accuracy can be at most $p = s + 1$.*
- *If s is even, the order of accuracy can be at most $p = s + 2$.*

(Note that consistent s-step methods explicitly have $2s - 1$ degrees of freedom.)

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(Note that consistent s-step methods explicitly have $2s - 1$ degrees of freedom.)

Theorem (Second Dahlquist Barrier)

No A-stable explicit multistep methods exist.

An implicit A-stable multistep method has order of accuracy at most $p = 2$.

Startup

$$\sum_{j=0}^{s} \alpha_j \boldsymbol{u}_{n+1-j} = k \sum_{j=0}^{s} \beta_j \boldsymbol{f}(t_{n+1-j}, \boldsymbol{u}_{n+1-j}),$$

How to start from $n = 0$ if $s > 1$?

Odds and ends for multi-step methods

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How to start from $n = 0$ if $s > 1$?

Usually accomplished with Runge-Kutta methods of similar order.

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Predictor-corrector methods

Explicit and implicit methods are frequently used in *predictor-corrector* frameworks, e.g.,:

- An explicit approximation to $\boldsymbol{u}_{n+1}$ is computed with an Adams-Bashforth method.
- This approximation is used as an emulator for the unknown $\boldsymbol{u}(t_{n+1})$ on the right-hand side of an Adams-Moulton method.

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Predictor-corrector methods are an example from a more general class of methods called *general linear methods*, which encompass both multi-stage and multi-step methods.

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