DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH

Analysis of Numerical Methods, II
MATH 6620 – Section 001 – Spring 2024
Homework 6
Fourier Series for time-dependent problems

Due Monday, April 22, 2024

Note: In this assignment, you must complete problems 1-3. However, you need only choose one of problems 4, 5, or 6 to complete and submit. (I.e., you must complete a total of 4 problems in this assignment.)

Corrections: corrections to an earlier version of this assignment are shown in blue boldface. You may choose to use the earlier, incorrect assignment, in which case you should justify what can be accomplished for those problems.

Submit your solutions online through Gradescope.

1. (Operator splitting)
Consider the following semi-discrete form of a PDE:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} = A(\boldsymbol{u}) + B(\boldsymbol{u}), \qquad \boldsymbol{u}(x,0) = \boldsymbol{u_0}(x).$$

where both A and B may be arbitrary operators.

a. Suppose A and B are linear operators (and hence $A(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u}$ and $B(\boldsymbol{u}) = \boldsymbol{B}\boldsymbol{u}$). Show that the numerical scheme,

$$\boldsymbol{u}_* = e^{k\boldsymbol{A}}\boldsymbol{u}_n$$
$$\boldsymbol{u}_{n+1} = e^{k\boldsymbol{B}}\boldsymbol{u}_*$$

is in general first-order accurate. Also give a sufficient condition on $\boldsymbol{A}$ and $\boldsymbol{B}$ to ensure that the scheme is second-order accurate.

b. Again assuming A and B are linear operators, show that the scheme

$$\boldsymbol{u}^* = e^{k\boldsymbol{A}/2}\boldsymbol{u}_n$$
$$\boldsymbol{u}^{**} = e^{k\boldsymbol{B}}\boldsymbol{u}_*$$
$$\boldsymbol{u}_{n+1} = e^{k\boldsymbol{A}/2}\boldsymbol{u}_{**}$$

is in general second-order accurate. (This scheme is called *Strang splitting.*)

c. Repeat the above procedures in the general nonlinear case, confirming that both schemes are first- and second-order accurate. I.e., replace $\boldsymbol{A}\boldsymbol{u}$ with $A(\boldsymbol{u})$, and similarly for the operator B, **and also replace exponential solution operators with exact solution operators for nonlinear problems**. (You need not give sufficient conditions on A and B to ensure that the first-order scheme is actually second-order accurate.)

2. (Fourier spectral methods)
Consider the partial differential equation,

$$u_t + (\sin x)\, u_x = \frac{1}{2}u_{xx}, \qquad u(x,0) = \sin x \tag{1}$$

for $u = u(x,t)$, $x \in [0, 2\pi)$, $t > 0$, and periodic boundary conditions.

a. Write out the semi-discrete form for the Fourier-Galerkin scheme for this problem.
b. If using Forward Euler for fully discretizing the semi-discrete form, what is the timestep restriction (say as a function of N) to ensure that the scheme is stable in the sense of absolute stability? (You may investigate this analytically or empirically via the region of stability.) What aspect of the equation (1) is the dominant contributor to the timestep restriction? How does this make the total computational time of the solver scale with N?
c. Implement both implicit (e.g., Crank-Nicolson) and explicit time-stepping methods (say RK2) for solving the above problem up to time $T = 1$. Visualize the simulated results and discuss the advantages and disadvantages of each approach.

3. (Viscous Burgers' equation)
Consider the viscous Burgers' partial differential equation,

$$u_t + f(u)_x = \nu u_{xx}, \quad x \in [0, 2\pi) \qquad f(u) = u^2, \qquad u(x, 0) = \sin x.$$

with periodic boundary conditions. Implement both a Fourier-Galerkin and Fourier-collocation solver for this equation. You may use an explicit time-stepping method.
a. For *viscous* Burgers', $\nu > 0$, show and discuss results for $t > 0$ when ν is small, and when ν is large.
b. For *inviscid* Burgers', $\nu = 0$, what differences do you observe between the collocation and Galerkin methods? Do the solutions appear to be accurate for large t?

Complete only one of problems 4-6, of your choice. For each problem: describe and write down a scheme (from the general types that we've covered in this class) to numerically solve the provided PDE. To whatever extent you are able, discuss consistency, stability, convergence, and computational cost of the scheme. Implement the scheme and discuss the numerical results (say with snapshots of the solution), investigating convergence when applicable. The terminal time T is not given; you should investigate the numerical solution for a reasonable range of t.

4. (Korteveg-De Vries equation) Consider the Korteweg-De Vries equation:

$$u_t + u_{xxx} - 6uu_x = 0, \qquad u(x, 0) = -2\text{sech}^2(x), \tag{2}$$

over the domain $x \in \mathbb{R}$. There are self-similar solitary wave solutions ("solitons") to this PDE. One such solution is:

$$u(x, t) = -2\text{sech}^2(x - 4t - 5)$$

There are also interacting soliton solutions. The following initial condition corresponds to one such solution:

$$u(x, 0) = -2\text{sech}^2(x - 5) - \frac{1}{2}\text{sech}^2\left(\frac{1}{2}(x - 1)\right).$$

5. (Multidimensional wave equation) Consider the partial differential equation

$$u_t + \boldsymbol{c} \cdot \nabla u(x, y) = 0,$$

with periodic boundary conditions on $(x,y) \in [0, 2\pi)^2$. Let the wavespeed be

$$\boldsymbol{c}(x,y) = (\sin(x+y), \sin(x-y)).$$

Let the initial condition be,

$$u(x,y,0) = \sin^6 x \ \sin^6 y$$

6. (1D Euler equations) Consider the one-dimensional Euler equations of gas dynamics. These equations, in conservation form, are written as

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0, \qquad\qquad t > 0,$$

where the conserved variable $\boldsymbol{u}(x,t) \in \mathbb{R}^3$ and flux $\boldsymbol{f}$ are functions of the physical gas density $\rho(x,t)$, momentum $m(x,t)$, and energy $E(x,t)$:

$$\boldsymbol{u} = \begin{bmatrix} \rho \\ m \\ E \end{bmatrix}, \qquad\qquad \boldsymbol{f} = \begin{bmatrix} m \\ p + \frac{m^2}{\rho} \\ (E+p)\frac{m}{\rho} \end{bmatrix},$$

where the pressure p and energy E are related by

$$p = (\gamma - 1)\left(E - \frac{m^2}{2\rho}\right),$$

where γ is the heat capacity ratio; for this problem set $\gamma = \frac{7}{5}$. The initial condition is,

$$\rho(x,0) = \begin{cases} \frac{5}{8} - \frac{3}{8}\sin\left(x - \frac{1}{2}\right), & \left|x - \frac{1}{2}\right| \leq \frac{\pi}{2} \\ 1, & x < \frac{1}{2} - \frac{\pi}{2} \\ \frac{1}{4}, & x > \frac{1}{2} + \frac{\pi}{2} \end{cases} \qquad\qquad m(x,0) = 0,$$

$$E(x,0) = \frac{1}{\gamma - 1}\begin{cases} \frac{3}{5} - \frac{2}{5}\sin\left(x - \frac{1}{2}\right), & \left|x - \frac{1}{2}\right| \leq \frac{\pi}{2} \\ 1, & x < \frac{1}{2} - \frac{\pi}{2} \\ \frac{1}{5}, & x > \frac{1}{2} + \frac{\pi}{2} \end{cases}$$

Consider the physical domain to be $x \in [-5, 5]$. The boundary conditions are Dirichlet conditions with values given by the value of the initial data at the boundaries.
The following may/not be useful: With $v := m/\rho$ the velocity variable, the flux Jacobian is,

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{3-\gamma}{2}v^2 & (3-\gamma)v & \gamma - 1 \\ -\frac{\gamma E v}{\rho} + (\gamma-1)v^3 & \frac{\gamma E}{\rho} - \frac{3(\gamma-1)v^2}{2} & \gamma v \end{bmatrix},$$

and is diagonalizable with real eigenvalues:

$$\lambda\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right) = \{v + c, \ v, \ v - c\},$$

where c is the (local) sound speed:

$$c = \sqrt{\frac{\gamma p}{\rho}}.$$