Department of Mathematics, University of Utah

Analysis of Numerical Methods, II
MATH 6620 – Section 001 – Spring 2024
Homework 5
Fourier Series and time-dependent problems

Due Friday, April 5, 2024

Submit your solutions online through Gradescope.

1. (Hyperbolic systems)

a. A linear PDE,

$$\boldsymbol{u}_t + \boldsymbol{A}\boldsymbol{u}_x = 0,$$

governing the vector-valued solution $\boldsymbol{u}(x,t) \in \mathbb{R}^m$, is called *hyperbolic* if $\boldsymbol{A}$ is diagonalizable and $\lambda(\boldsymbol{A}) \subset \mathbb{R}$, with $\lambda(\boldsymbol{A})$ denoting the spectrum of $\boldsymbol{A}$. Assuming this PDE is hyperbolic, and using an equispaced grid in time and space, derive an implementable version of the *upwind scheme* for this problem, where $\boldsymbol{u}_t$ is discretized as $D^+\boldsymbol{u}_j^n$. You may ignore boundary conditions.

b. Derive and state a CFL condition for your upwind scheme.

c. Consider a *nonlinear* PDE,

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0,$$

where $\boldsymbol{f} : \mathbb{R}^m \to \mathbb{R}^m$, and again we ignore boundary conditions. Propose a definition of a *hyperbolic PDE* in this context, and identify an upwind-like numerical scheme on an equidistant time and space mesh. (Again, discretize $\boldsymbol{u}_t \approx D^+\boldsymbol{u}_j^n$.) What is the CFL condition for your scheme?

2. (Hyperbolic systems in two spatial dimensions) Consider a linear system of PDEs of the form,

$$\boldsymbol{u}_t + \boldsymbol{A}\boldsymbol{u}_x + \boldsymbol{B}\boldsymbol{u}_y = 0,$$

for the unknown $\boldsymbol{u}(x,y,t) \in \mathbb{R}^m$, where $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m\times m}$ are given. Assume this system is *hyperbolic*, meaning that for every $\alpha, \beta \in \mathbb{R}$, then $\alpha\boldsymbol{A}+\beta\boldsymbol{B}$ is diagonalizable with $\lambda(\alpha\boldsymbol{A}+\beta\boldsymbol{B}) \subset \mathbb{R}$. We will use the abbreviation $\boldsymbol{x} = (x,y)^T \in \mathbb{R}^2$.

a. Determine plane wave solutions: let $\boldsymbol{n} \in \mathbb{R}^2$ satisfying $\|\boldsymbol{n}\|_2 = 1$ be a given vector, and assume some initial data $\boldsymbol{u}(x,y,0) = \boldsymbol{u}_0(\boldsymbol{n}\cdot\boldsymbol{x})$. (Note that $\boldsymbol{u}_0$ is a function of a scalar input.) Ignoring boundary conditions, determine the solution $\boldsymbol{u}(x,y,t)$.

b. Assume further that $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric matrices. Consider discretizing this PDE on an equidistant spatial grid on a square: $h_x > 0$ and $h_y > 0$ are the grid spacing in the x- and y-directions, respectively. Again, assume the discretization $\boldsymbol{u}_t \approx D^+\boldsymbol{u}_{i,j}^n$, where $\boldsymbol{u}_{i,j}^n \approx u(x_i, y_j, t_n)$. Using plane waves as motivation, derive a CFL condition (a condition on the timestep k) for such a discretization. You may assume that spatial derivatives are approximated using the stencil involving $u_{i\pm1,j\pm1}^n$ and that the numerical domain of dependence is the convex hull of the stencil points (as in the one-dimensional case).

3. (Finite differences for non-smooth problems) Consider Burgers' equation:

$$u_t + f(u)_x = 0, \qquad f(u) = \frac{u^2}{2}, \qquad (x,t) \in [-\pi, \pi] \times (0, T].$$

where we will take $T = 1.0$. Supplement this PDE with the boundary conditions,

$$u(\pm\pi, t) = u(\pm\pi, 0),$$

where the initial condition function $u(\cdot, 0)$ will be specified below. Note that for smooth u, then $f(u)_x = uu_x$. Based on this observation, and using an equidistant grid in both space and time, we will consider two schemes for this PDE:

$$\begin{aligned} \text{Scheme } A: &\quad D^+ u_j^n + D_0 f(u_j^n) = 0, \\ \text{Scheme } B: &\quad D^+ u_j^n + u_j^n D_0 u_j^n = 0. \end{aligned}$$

Numerically test these schemes for solving the PDE up to time $t = T$ with the following three initial data:

$$u(x,0) = u_1(x) = -\sin(x).$$

$$u(x,0) = u_2(x) = \begin{cases} 1, & x \le 0 \\ 0, & x > 0 \end{cases}$$

$$u(x,0) = u_3(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$$

Based on the experiments above, which scheme would you prefer to use for each example, and in general? (Feel free to try other schemes as well, e.g., upwind versions of Schemes A and B as identified in problem 1c.)

4. (Fourier approximation)

a. Prove that if $u \in H_p^s$ for some integer $s \ge 0$, then

$$\|u - P_N u\|_{L^2} \le N^{-s} \|u\|_{H_p^s},$$

where the space H_p^s is defined on slide `D13-S10` and P_N is defined in slide `D13-S07`.

b. Confirm this behavior by numerically computing $\|u_j - P_N u_j\|_{L^2}$ as a function of N for each $j = 0, 1, 2, 3$. The functions u_j, $j \ge 0$, are defined as,

$$u_0(x) := \begin{cases} 1, & |x - \pi| < \frac{\pi}{2} \\ -1, & \text{else} \end{cases} \qquad u_j(x) := c_j + \int_0^x u_{j-1}(y)\, \mathrm{d}y \quad (j \ge 1),$$

where c_j is chosen so that u_j is a mean-0 function. Based on your numerical results what type of regularity (s) does u_j seem to have?

5. (Fourier interpolation)

a. For any $N \ge 1, k \in \mathbb{Z}$, prove that $I_N \phi_k = \phi_\ell$, where ℓ satisfying $|\ell| \le N$ is the modular restriction of k to $[-N, N]$:

$$\ell = \ell(k) := k - (2N+1)j \in [-N, N], \qquad j \in \mathbb{Z}.$$

An equivalent definition: $\ell(k) = -N + [(k + N) \pmod{2N+1}]$. The operator I_N is defined on `D14-S10` as the $M = (2N+1)$-point Fourier interpolation operator, and ϕ_k is defined on slide `D13-S03(b)`.

b. If $u \in H_p^s$, prove that,

$$\|u - I_N u\|_{L^2} \lesssim N^{-s} \|u\|_{H_p^s},$$

where $a \lesssim b$ means that $a \le Cb$ for some constant C independent of N and u.