Department of Mathematics, University of Utah

Analysis of Numerical Methods, II

MATH 6620 – Section 001 – Spring 2024

Homework 4

Finite difference methods for time-dependent problems

Due Friday, March 22, 2024

Submit your solutions online through Gradescope.

1. (Stability)

a. Compute the region of the stability for the leapfrog method:

$$D^0 u^n = f^n,$$

for the ODE $u' = f(t,u)$, with $u^n \approx u(t^n)$ and $f^n \approx f(t^n, u(t^n))$.

b. Investigate and discuss the consistency, stability, and practicality of the scheme,

$$D^0 u_j^n = D_+ D_- u_j^n,$$

for the PDE $u_t = u_{xx}$.

c. Use von Neumann analysis to determine stability requirements for the Lax-Friedrichs and Lax-Wendroff schemes for $u_t + a u_x = 0$, which are, respectively,

$$D^+ u_j^n = -a D_0 u_j^n + \frac{h^2}{2k} D_+ D_- u_j^n$$
$$D^+ u_j^n = -a D_0 u_j^n + \frac{a^2 k}{2} D_+ D_- u_j^n$$

2. (Semi-Lagrangian methods) Consider the PDE,

$$u_t + a u_x = 0, \qquad u(x,0) = u_0(x), \qquad a > 0,$$

with periodic boundary conditions for $x \in [0, 2\pi)$. Use our standard notation, $u_j^n \approx u(x_j, t^n)$ with $x_j = jh$ and $t^n = nk$ with $h = 2\pi/M$ and $k > 0$. In this problem, we will investgiate particular *semi-Lagrangian* schemes. Of particular use will be characteristic curves. Let $X(t; x_j, t^n)$ denote the characteristic curve passing through at $x = x_j$ at time $t = t^n$.

a. Consider the following scheme: Let the values of u_j^n for all j be given. We compute u_j^{n+1} as,

$$u_j^{n+1} = p_j(y), \qquad y := X(t^n; x_j, t^{n+1})$$

where p_j is the linear interpolant formed by the data (x_{j-1}, u_{j-1}^n) and (x_j, u_j^n). Show that this scheme is equivalent to the upwind scheme, and give conditions on a, k, h that ensure that $p_j(y)$ is an interpolatory evaluation instead of an extrapolatory evaluation.

b. Consider the scheme,

$$u_j^{n+1} = q_j(y), \qquad y := X(t^n; x_j, t^{n+1}),$$

where q is the quadratic interpolant formed from (x_{j-1}, u_{j-1}^n), (x_j, u_j^n), and (x_{j+1}, u_{j+1}^n). Explicitly write the scheme as a function of $\{u_j^n\}_j$ for this method, and compare against other schemes for this PDE that we've discussed in class.

---

3. (Orders of convergence)
This problem provides exercises to numerically confirm expected orders of convergence.

a. For the PDE,

$$u_t = u_{xx}, \qquad u(x,0) = \exp(\sin x), \qquad (x,t) \in [0, 2\pi) \times (0, T],$$

with periodic boundary conditions in x, numerically confirm that the scheme,

$$D^+ u_j^n = \frac{1}{2} D_+ D_- u_j^n + \frac{1}{2} D_+ D_- u_j^{n+1},$$

is second-order in space and time, say for $T = 1$.

b. For the PDE,

$$u_t = -u_x, \qquad u(x,0) = \exp(\sin x), \qquad (x,t) \in [0, 2\pi) \times (0, T],$$

with periodic boundary conditions in x, numerically confirm that the scheme,

$$D^0 u_j^n = -D_0 u_n^j,$$

is second-order in space and time, say for $T = 1$.