Department of Mathematics, University of Utah

Analysis of Numerical Methods, II

MATH 6620 – Section 001 – Spring 2024

Homework 3

Time-stepping methods, II

Due Wednesday, February 28, 2024

Submit your solutions online through Gradescope.

1. (Runga-Kutta Methods)

a. Recall Ralston's method from the previous assignment:

$$\boldsymbol{u}_{n+1} = \boldsymbol{u}_n + \frac{k}{4}\boldsymbol{f}(t_n, \boldsymbol{u}_n) + \frac{3k}{4}\boldsymbol{f}\left(t_n + \frac{2}{3}k, \boldsymbol{u}_n + \frac{2}{3}k\boldsymbol{f}(t_n, \boldsymbol{u}_n)\right),$$

Identify the Butcher tableau for this method.

b. Show that Ralston's method is consistent to second order.

2. (Multi-step methods)

a. Compute coefficients for the following implicit multi-step scheme that achieves the optimal order of accuracy,

$$\boldsymbol{u}_{n+1} + \alpha_1 \boldsymbol{u}_n + \alpha_2 \boldsymbol{u}_{n-1} = k\beta_0 \boldsymbol{f}_{n+1} + k\beta_1 \boldsymbol{f}_n + k\beta_2 \boldsymbol{f}_{n-1},$$

where $\boldsymbol{f}_j \coloneqq \boldsymbol{f}(t_j, \boldsymbol{u}_j)$.

b. Identify the order of consistency of the scheme, and determine whether this method is 0-stable and/or A-stable.

3. (SSP Methods)

In this problem, consider an autonomous ODE, $\boldsymbol{u}' = \boldsymbol{f}(\boldsymbol{u})$.

a. Consider an s-stage explicit Runge-Kutta method. For each $m = 2, \ldots, s+1$, let constants $\{\alpha_{m,j}\}_{j=1}^{m-1}$ be given such that $\alpha_{m,j} \geq 0$ and $\sum_{j=1}^{m-1} \alpha_{m,j} = 1$. Show that such an s-stage explicit method can be written as,

$$\begin{aligned} \boldsymbol{U}_1 &\coloneqq \boldsymbol{u}_n, \\ \boldsymbol{U}_m &\coloneqq \sum_{j=1}^{m-1} \left(\alpha_{m,j}\boldsymbol{U}_j + \beta_{m,j}\boldsymbol{f}(\boldsymbol{U}_j)\right) \qquad\qquad 2 \leq m \leq s+1 \\ \boldsymbol{u}_{n+1} &= \boldsymbol{U}_{s+1} \end{aligned}$$

b. Let $|\cdot|$ be any seminorm on vectors $\boldsymbol{u}$, and suppose that there exists a $k_* > 0$ such that for all $\boldsymbol{u}$ and $k \in (0, k_*]$, then $|\boldsymbol{u} + k\boldsymbol{f}(\boldsymbol{u})| \leq |\boldsymbol{u}|$. Assume that the $\alpha_{m,j}$ coefficients above can be chosen so that $\beta_{m,j} \geq 0$ for all j, m. Show that there is a $c > 0$ such that

$$k \in (0, ck_*] \implies |\boldsymbol{u}_{n+1}| \leq |\boldsymbol{u}_n|$$

and explicitly identify a formula for c in terms of the $\alpha_{m,j}$ and $\beta_{m,j}$. Schemes that satisfy this are called (Runge-Kutta) *Strong Stability Preserving* (SSP) schemes. The constant c is called the *SSP coefficient*. (The point here is that it's somewhat easy to establish boundedness of the seminorm $|\cdot|$ for a simple Forward Euler scheme; SSP methods allow one to directly port this boundedness to higher order methods.)

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c. Verify that the following is an SSP scheme:

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \hline & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{array}$$

d. Is the Ralston method from problem 1 an SSP scheme? If so, compute its SSP coefficient.

4. (Exponential Integrators)
For this problem, consider the ODE,

$$\boldsymbol{u}'(t) = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{N}(t, \boldsymbol{u}),$$

where $\boldsymbol{A}$ is a fixed matrix and $\boldsymbol{N}$ is an arbitrary, e.g., nonlinear, function.

a. With initial data $\boldsymbol{u}(0) = \boldsymbol{u}_0$, show that the solution to this IVP at time $t > 0$ is given by,

$$\boldsymbol{u}(t) = e^{t\boldsymbol{A}}\boldsymbol{u}_0 + \int_0^t e^{(t-s)\boldsymbol{A}}\boldsymbol{N}(s, \boldsymbol{u}(s))\,\mathrm{d}s, \tag{1}$$

where $e^{t\boldsymbol{A}}$ is the matrix exponential of $t\boldsymbol{A}$.

b. *Exponential Integrators* form a scheme by setting $(0, t) \leftarrow (t_n, t_{n+1})$, replacing $e^{t\boldsymbol{A}}$ with $e^{(t_{n+1}-t_n)\boldsymbol{A}}$, and discretizing the integral above by approximating $\boldsymbol{N}(\boldsymbol{u}(s))$ with a quadrature rule/polynomial approximation. The matrix exponential term is treated (integrated) exactly. For example, Forward Euler makes the approximation $\boldsymbol{N}(\boldsymbol{u}(s)) \approx \boldsymbol{N}(\boldsymbol{u}_n)$. In terms of matrix operations (possibly including the matrix exponential) write out the Forward Euler and explicit midpoint (RK2) exponential integrator schemes. The explicit midpoint ("modified Euler") scheme has the tableau,

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array}$$

c. Use both exponential integrator schemes to numerically solve,

$$u_t = u_{xx} + 100u^6(1 - u^6), \qquad u(x, 0) = \sin \pi x + \frac{1}{4}\sin 2\pi x,$$

with a finite-difference scheme in space for $x \in [0, 1]$ with boundary conditions $u(0) = u(1) = 1$ up to terminal time $T = 1$. Numerically investigate the k-order of convergence of these schemes.

5. (Well-posed linear PDEs)
Consider the IVP,

$$u_t = 3u_x - u_{xx} - u_{xxxx}, \qquad u(x, 0) = u_0(x), \tag{2}$$

with periodic boundary conditions on $x \in [0, 2\pi)$.

a. Determine if the PDE is well-posed in the sense of the definition on slide `D10-S05`.

b. Compute the exact solution to this PDE.