Department of Mathematics, University of Utah
Analysis of Numerical Methods, II
MATH 6620 – Section 001 – Spring 2024
Homework 2
Time-stepping methods, I

Due Friday, February 9, 2024

Submit your solutions online through Gradescope.

1. (Quadrature rules)

a. Show that

$$\frac{1}{h}\int_0^h u(x)\,\mathrm{d}x \approx u\left(\frac{h}{2}\right),$$

is a second-order approximation in h.

b. Compute (polynomial) interpolatory quadrature weights for the rule,

$$\int_0^1 f(x)\,\mathrm{d}x \approx w_1 f(0) + w_2 f\left(\frac{1}{3}\right) + w_3 f(1)$$

What is the polynomial degree of exactness for this rule?

c. Compute quadrature weights for the rule,

$$f'(0) + \int_{-1}^1 f(x)\,\mathrm{d}x \approx w_{-1} f(-1) + w_0 f(0) + w_1 f(1).$$

that is exact for all quadratic polynomials.

2. (FD convergence)
For the boundary value problem,

$$-u''(x) = f(x), \qquad\qquad u(0) = u(1) = 0,$$

with f given, consider the standard 3-point stencil finite-difference approximation,

$$-D_+ D_- u_j = f_j, \qquad\qquad j \in [M],$$

where for a given $M \in \mathbb{N}$,

$$u_j \approx u(x_j), \qquad\qquad x_j \coloneqq jh, \qquad\qquad h \coloneqq \frac{1}{M+1},$$

with $u_0 = u_{M+1} = 0$. Prove that in the scaled ℓ^1 norm,

$$\|\boldsymbol{u}\|_{1,h} \coloneqq h \sum_{j\in[M]} |u_j| \approx \int_0^1 |u(x)|\,\mathrm{d}x,$$

that this scheme is convergent to second order in h. You may cite without proof any results given in class/on the slides.

3. (ODE convergence)
For the ODE system $\boldsymbol{u}'(t) = \boldsymbol{f}(t, \boldsymbol{u})$ with initial condition $\boldsymbol{u}(0)$ given and $\boldsymbol{f}$ globally Lipschitz continuous in $\boldsymbol{u} \in \mathbb{R}^M$ uniformly in t, show that backward Euler with the initial state $\boldsymbol{u}_0 = \boldsymbol{u}(0)$ is *convergent* to first order as defined on slide `D06-S07`.

4. (Regions of stability)

a. Compute the stability/amplification factor for Crank-Nicolson and show that it is the left half-plane.

b. Compute the stability/amplification factor for the following scheme,

$$\boldsymbol{u}_{n+1} = \boldsymbol{u}_n + \frac{k}{4}\boldsymbol{f}(t_n, \boldsymbol{u}_n) + \frac{3k}{4}\boldsymbol{f}\left(t_n + \frac{2}{3}k, \boldsymbol{u}_n + \frac{2}{3}k\boldsymbol{f}(t_n, \boldsymbol{u}_n)\right),$$

and plot the corresponding stability region, using software if desired. (This scheme is called Ralston's method.)

5. (ODE solvers)
Consider the IVP,

$$\boldsymbol{u}'(t) = \boldsymbol{A}\boldsymbol{u}, \qquad (\boldsymbol{u}(0))_j = j(-1)^j, \qquad \boldsymbol{A} = C^2 \begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2 \end{pmatrix} \in \mathbb{R}^{10\times 10} \tag{1}$$

where $j \in [10]$, $\boldsymbol{A}$ is symmetric and tridiagonal, and $C \in \mathbb{R}$. We'll compute numerical solutions in this problem up to a terminal time $T = 1$.

a. Take $C = 3$. Implement both Forward Euler and Crank-Nicolson solvers for this IVP, and demonstrate that the schemes exhibit the expected orders of convergence (with respect to the time discretization parameter k) for each method. (You may use any norm on $\boldsymbol{u}$.) Briefly discuss the advantages and disadvantages of each scheme.

b. Take $C = 10$. Again discuss the advantages and disadvantages of each scheme, using numerical results to support your conclusions.