Department of Mathematics, University of Utah

Analysis of Numerical Methods, II

MATH 6620 – Section 001 – Spring 2024

Homework 1

Finite differences for 1D stationary problems

Due Friday, Jan 26, 2024

Submit your solutions online through Gradescope.

1. (Finite difference formulas)

In the following, fix $M \in \mathbb{N}$, define $h = 1/(M+1)$, and let $x_j = jh$ for $j = 0, \ldots, M+1$ be an equidistant grid on $[0, 1]$. Let $U_j \coloneqq u(x_j)$ denote the value of a function u at x_j.

- **a.** Compute a three-point one-sided finite difference approximation to the second derivative of u at x_0, and identify the order of accuracy of this approximation. I.e., use U_0, U_1, U_2 to compute an approximation to $u''|_{x_0}$.
- **b.** Use a centered five-point stencil of nearest neighbors to compute an approximation to u'' at x_j (for $2 \leq j \leq M-1$), accurate to as high an order of approximation as possible. What is the order of accuracy of this approximation?
- **c.** Consider $D_0 D_0 U_j$. What quantity does this approximate, and to what order?

2. (Finite difference methods in 1D)

For the ODE,

$$-u''(x) = f(x), \qquad x \in (0, 1), \tag{1}$$

with homogeneous boundary conditions $u(0) = u(1) = 0$, empirically confirm that the numerical scheme

$$-D_+ D_- u_j = f_j, \qquad j \in [M],$$

is second-order convergent. Here, $f_j = f(x_j)$ and $u_j \approx u(x_j)$, with

$$x_j = jh, \qquad h = \frac{1}{M+1},$$

for some $M \in \mathbb{N}$. For this example, use $f(x) = -2\pi \cos \pi x + \pi^2 x \sin \pi x$, for which the exact solution is $u(x) = x \sin \pi x$. To "confirm" the order of convergence, plot the scaled error,

$$\|\boldsymbol{u} - \boldsymbol{U}\|_{2,h} \coloneqq \sqrt{h} \|\boldsymbol{u} - \boldsymbol{U}\|_2,$$

as a function of h on a log-log plot, and visually compare the data to a line of slope 2. Above, $U_j = u(x_j)$. In your solution, explicitly write all these details, so that the solution is readable by a person who would not have read the problem statement.

3. (Finite difference methods in 1D)

Consider the ordinary differential equation:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\kappa(x) \frac{\mathrm{d}}{\mathrm{d}x} u(x)\right) = f(x), \qquad x \in (0, 1), \tag{2}$$

with homogeneous Dirichlet boundary conditions, $u(0) = u(1) = 0$, and where the scalar diffusion coefficient κ is given by,

$$\kappa(x) = 2 + \sum_{\ell=1}^{5} \frac{1}{\ell+1} \sin(\ell \pi x).$$

The goal of this exercise will be to numerically compute solutions to this problem.

a. Define the operator,

$$\widetilde{D}_0 u(x_j) = \frac{u(x_j + h/2) - u(x_j - h/2)}{h}, \qquad h = 1/(M+1), \qquad x_j \coloneqq jh,$$

for a fixed number of points $M \in \mathbb{N}$. Then with u_j the numerical solution approximating $u(x_j)$, consider the scheme,

$$-\widetilde{D}_0 \left(\kappa(x_j) \widetilde{D}_0 u_j \right) = f(x_j), \qquad j \in [M]. \tag{3}$$

Show that, for smooth u and κ, this scheme has second-order local truncation error.

b. Construct an exact solution via the *method of manufactured solutions*: posit an exact (smooth) nontrivial solution $u(x)$ (that satisfies the boundary conditions!) and compute f in (2) so that your posited solution satisfies (2).

c. Implement the scheme above for solving (2), setting f to be the function identified in part (b), so that you know the exact solution. Show that indeed you achieve second-order convergence in h (in the $h^{1/2}$-scaled vector ℓ^2 norm as in the previous problem).