Department of Mathematics, University of Utah

Analysis of Numerical Methods, II

MATH 6620 – Section 001 – Spring 2024

Homework 1 Solutions

Finite differences for 1D stationary problems

Due Friday, Jan 26, 2024

Submit your solutions online through Gradescope.

1. (Finite difference formulas)

In the following, fix $M \in \mathbb{N}$, define $h = 1/(M+1)$, and let $x_j = jh$ for $j = 0, \ldots, M+1$ be an equidistant grid on $[0,1]$. Let $U_j \coloneqq u(x_j)$ denote the value of a function u at x_j.

- **a.** Compute a three-point one-sided finite difference approximation to the second derivative of u at x_0, and identify the order of accuracy of this approximation. I.e., use U_0, U_1, U_2 to compute an approximation to $u''|_{x_0}$.
- **b.** Use a centered five-point stencil of nearest neighbors to compute an approximation to u'' at x_j (for $2 \le j \le M-1$), accurate to as high an order of approximation as possible. What is the order of accuracy of this approximation?
- **c.** Consider $D_0 D_0 U_j$. What quantity does this approximate, and to what order?

<u>**Solution**</u>:

a. We seek constants A, B, C such that,

$$AU_0 + BU_1 + CU_2 \approx u''(x_0)$$

To accomplish this, we note that $x_j = x_0 + jh$, and use the following Taylor Series approximations,

$$\begin{array}{rcccccccccccc} U(x_1) & = & U(x_0+h) & = & U_0 & + & hU_0' & + & \frac{h^2}{2}U_0'' & + & \frac{h^3}{6}U_0''' & + & \mathcal{O}(h^4) \\ U(x_2) & = & U(x_0+2h) & = & U_0 & + & 2hU_0' & + & 2h^2U_0'' & + & \frac{4h^3}{3}U_0''' & + & \mathcal{O}(h^4) \end{array}$$

where $U_j' = u'(x_j)$, and similarly for U_j'', U_j'''. Therefore, we have:

$$\begin{aligned} AU_0 + BU_1 + CU_2 = {} & U_0\left[A+B+C\right] + \\ & + U_0'\left[Bh + 2hC\right] + \\ & + U_0''\left[B\frac{h^2}{2} + 2Ch^2\right] + \\ & + U_0'''\left[B\frac{h^3}{6} + \frac{4h^3}{3}C\right] + \mathcal{O}(h^4). \end{aligned}$$

In order for the right-hand side to approximate U_0'', we require the following conditions on the first three bracketed terms involving A, B, C:

$$\begin{aligned} A + B + C &= 0 \\ B + 2C &= 0 \\ \frac{B}{2} + 2C &= \frac{1}{h^2}. \end{aligned}$$

The solution to this 3×3 linear system is,

$$(A, B, C) = \frac{1}{h^2}(1, -2, 1).$$

Therefore, the desired approximation is

$$U_0'' \approx \frac{1}{h^2}U_0 - \frac{2}{h^2}U_1 + \frac{1}{h^2}U_2.$$

This is a first-order accurate approximation since we have:

$$\begin{aligned} AU_0 + BU_1 + CU_2 &= U_0'' + U_0''' \left[B\frac{h^3}{6} + \frac{4h^3}{3}C \right] + \mathcal{O}(h^2) \\ &= U_0'' + U_0''' \left[-\frac{h}{3} + \frac{4h}{3} \right] + \mathcal{O}(h^2) \\ &= U_0'' + \mathcal{O}(h) \end{aligned}$$

b. To construct a five-point approximation to the second derivative at x_j, we again compile some Taylor Series:

$$\begin{aligned} U(x_{j\pm1}) &= U(x_j \pm h) = U_j \pm hU_j' + \tfrac{h^2}{2}U_j'' \pm \tfrac{h^3}{6}U_j''' + \tfrac{h^4}{24}U_j'''' \pm \tfrac{h^5}{120}U_j^{(5)} + \tfrac{h^6}{720}U_j^{(6)} + \cdots \\ U(x_{j\pm2}) &= U(x_j \pm 2h) = U_0 \pm 2hU_j' + 2h^2U_j'' \pm \tfrac{4h^3}{3}U_0''' + \tfrac{2}{3}h^4U_0'''' \pm \tfrac{4}{15}h^5U_0^{(5)} + \tfrac{4}{45}h^6U_j^{(6)} + \cdots \end{aligned}$$

Once again we attempt to find coefficients (A, B, C, D, E) such that,

$$AU_{j-2} + BU_{j-1} + CU_j + DU_{j+1} + EU_{j+2} \approx U_0'',$$

which when using our Taylor Series approximations leads to the constraints,

$$\begin{aligned} A + B + C + D + E &= 0 \\ -2A - B + D + 2E &= 0 \\ 2A + \frac{B}{2} + \frac{D}{2} + 2E &= \frac{1}{h^2} \\ -\frac{4}{3}A - \frac{1}{6}B + \frac{1}{6}D + \frac{4}{3}E &= 0 \\ \frac{2}{3}A + \frac{1}{24}B + \frac{1}{24}D + \frac{2}{3}E &= 0 \end{aligned}$$

The second and fourth equations imply $A = E$ and $B = D$, so the 5×5 system above reduces to the following 3×3 system for (A, B, C):

$$\begin{aligned} 2A + 2B + C &= 0 \\ 4A + B &= \frac{1}{h^2} \\ \frac{4}{3}A + \frac{1}{12}B &= 0, \end{aligned}$$

from which we derive the solution:

$$(A, B, C, D, E) = \frac{1}{h^2}\left(-\frac{1}{12}, \frac{16}{12}, -\frac{30}{12}, \frac{16}{12}, -\frac{1}{12} \right).$$

c. First we compute the stencil and coefficients for this approximation:

$$\begin{aligned} D_0 D_0 U_j = D_0 \left(\frac{U_{j+1} - U_{j-1}}{2h} \right) &= \frac{1}{2h} D_0 U_{j+1} - \frac{1}{2h} D_0 U_{j-1} \\ &= \frac{1}{2h} \left(\frac{U_{j+2} - U_j}{2h} \right) - \frac{1}{2h} \left(\frac{U_j - U_{j-2}}{2h} \right) \\ &= \frac{1}{(2h)^2} \left(U_{j-2} - 2U_j + U_{j+2} \right). \end{aligned}$$

The above is simply the standard 3-point centered approximation to the second derivative with a mesh spacing of $2h$, which has order of accuracy $\mathcal{O}((2h)^2) = \mathcal{O}(h^2)$. To confirm this, we have already seen that the Taylor approximation is,

$$u(x \pm 2h) = u(x) \pm 2hu'(x) + 2h^2 u''(x) \pm \frac{4}{3} h^3 u'''(x) + \frac{2}{3} h^4 u^{(4)}(x) + \cdots$$

And therefore,

$$\frac{1}{(2h)^2} \left(U_{j-2} - 2U_j + U_{j+2} \right) = u''(x) + \frac{1}{3} h^2 u^{(4)}(x) + \cdots = u''(x) + \mathcal{O}(h^2).$$

2. (Finite difference methods in 1D)
For the ODE,

$$-u''(x) = f(x), \qquad x \in (0,1), \tag{1}$$

with homogeneous boundary conditions $u(0) = u(1) = 0$, empirically confirm that the numerical scheme

$$-D_+ D_- u_j = f_j, \qquad j \in [M],$$

is second-order convergent. Here, $f_j = f(x_j)$ and $u_j \approx u(x_j)$, with

$$x_j = jh, \qquad h = \frac{1}{M+1},$$

for some $M \in \mathbb{N}$. For this example, use $f(x) = -2\pi \cos \pi x + \pi^2 x \sin \pi x$, for which the exact solution is $u(x) = x \sin \pi x$. To "confirm" the order of convergence, plot the scaled error,

$$\|\boldsymbol{u} - \boldsymbol{U}\|_{2,h} \coloneqq \sqrt{h} \|\boldsymbol{u} - \boldsymbol{U}\|_2,$$

as a function of h on a log-log plot, and visually compare the data to a line of slope 2. Above, $U_j = u(x_j)$. In your solution, explicitly write all these details, so that the solution is readable by a person who would not have read the problem statement.

<u>Solution</u>: To approximate the solution u to the ODE

$$-u''(x) = f(x), \qquad x \in (0,1),$$

with homogeneous boundary conditions $u(0) = u(1) = 0$, we implement the scheme

$$-D_+ D_- u_j = f_j, \qquad j \in [M],$$

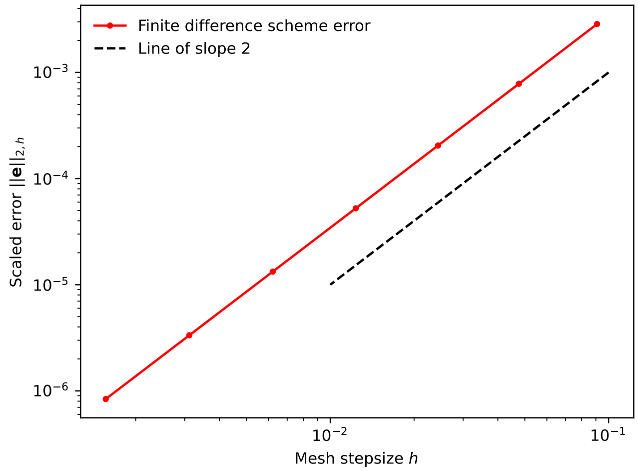

Figure 1: Numerical results for problem 2: second-order convergence of a 3-point central finite-difference approximation in 1D.

where $f_j = f(x_j)$,

$$x_j = jh, \qquad\qquad h = \frac{1}{M+1},$$

to compute $\boldsymbol{u} = (u_1, \ldots, u_M)^T$, where $u_j \approx u(x_j)$. Since the local truncation error for this scheme is $\mathcal{O}(h^2)$, we expect second-order convergence for the error, defined as,

$$\|\boldsymbol{e}\|_{2,h} := \sqrt{h}\sqrt{\sum_{j=1}^{M}(u_j - u(x_j))^2}.$$

To test this scheme, we use the following forcing function f and its corresponding solution u:

$$f(x) = -2\pi\cos\pi x + \pi^2 x \sin\pi x, \qquad\qquad u(x) = x\sin\pi x.$$

The results for various choices of h are shown in Figure 1, confirming with empirical results that the scheme exhibits second-order convergence.

3. (Finite difference methods in 1D)
Consider the ordinary differential equation:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\kappa(x)\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right) = f(x), \qquad\qquad x \in (0,1), \tag{2}$$

with homogeneous Dirichlet boundary conditions, $u(0) = u(1) = 0$, and where the scalar diffusion coefficient κ is given by,

$$\kappa(x) = 2 + \sum_{\ell=1}^{5}\frac{1}{\ell+1}\sin(\ell\pi x).$$

The goal of this exercise will be to numerically compute solutions to this problem.

a. Define the operator,

$$\widetilde{D}_0 u(x_j) = \frac{u(x_j + h/2) - u(x_j - h/2)}{h}, \qquad h = 1/(M+1), \qquad x_j \coloneqq jh,$$

for a fixed number of points $M \in \mathbb{N}$. Then with u_j the numerical solution approximating $u(x_j)$, consider the scheme,

$$-\widetilde{D}_0 \left(\kappa(x_j) \widetilde{D}_0 u_j \right) = f(x_j), \qquad j \in [M]. \tag{3}$$

Show that, for smooth u and κ, this scheme has second-order local truncation error.

b. Construct an exact solution via the *method of manufactured solutions*: posit an exact (smooth) nontrivial solution $u(x)$ (that satisfies the boundary conditions!) and compute f in (2) so that your posited solution satisfies (2).

c. Implement the scheme above for solving (2), setting f to be the function identified in part (b), so that you know the exact solution. Show that indeed you achieve second-order convergence in h (in the $h^{1/2}$-scaled vector ℓ^2 norm as in the previous problem).

<u>**Solution**</u>:

a. Let $\kappa_j = \kappa(x_j)$ and $U_j \coloneqq u(x_j)$. Then,

$$\widetilde{D}_0 U_j = U_j' + \frac{h^2}{24} U_j''' + \mathcal{O}(h^4).$$

Then since κ and u are smooth:

$$\begin{aligned}
-\widetilde{D}_0 \left(\kappa_j \widetilde{D}_0 U_j \right) &= -\widetilde{D}_0 \left(\kappa_j U_j' + \frac{h^2}{24} \kappa_j U_j''' + \mathcal{O}(h^4) \right) \\
&= -\left(\kappa_j U_j' \right)' + \frac{h^2}{24} \left(\kappa_j U_j' \right)''' + \mathcal{O}(h^4) - \frac{h^2}{24} \left(\kappa_j U_j''' \right)' + \mathcal{O}(h^4) + \mathcal{O}(h^4) \\
&= -\left(\kappa_j U_j' \right)' + \mathcal{O}(h^2) \\
&\overset{(2)}{=} f_j + \mathcal{O}(h^2).
\end{aligned}$$

Hence, the local truncation error, i.e., the residual of (3) with $u_j \leftarrow U_j$ is:

$$LTE = -\widetilde{D}_0 \left(\kappa_j \widetilde{D}_0 U_j \right) - f(x_j) = \mathcal{O}(h^2).$$

b. There are many possible choices. We'll consider $u(x) = \sin \pi x$, for which we have,

$$f(x) = -\left(\kappa(x) u' \right)' = -\pi \left(\kappa(x) \cos \pi x \right)' = -\pi \kappa'(x) \cos \pi x + \pi^2 \kappa(x) \sin \pi x.$$

c. We use a similar procedure as in problem 2 to compute the error. This results in the very visually similar results in Figure 2, again confirming second-order convergence for this finite difference scheme for non-constant diffusion coefficient.

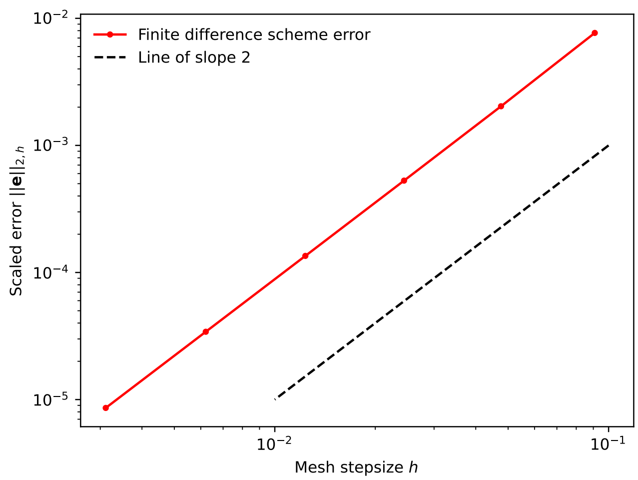

Figure 2: Numerical results for problem 3: second-order convergence of a 3-point central finite-difference approximation in 1D with non-constant diffusion coefficient.