Math 5760/6890: Introduction to Mathematical Finance

Geometric Brownian Motion and SDE's

See Petters and Dong 2016, Section 6.7-6.8

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We have introduced the stochastic process Brownian motion, $B_t = B(t)$.

- $P(B(\overset{0}{\cancel{t}}) = 0) = 1$
- B is continuous with probability 1
- The n sequential increments formed by any choice of $n+1$ ordered time points $t_1, \ldots, t_{n+1}$ are mutually independent
- For any $0 \leqslant s \leqslant t < \infty$, then $B(t) - B(s) \sim \mathcal{N}(0, t-s)$.

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Using this, we have defined the Itô integral:

$$\int_0^T f(t)\mathrm{d}B_t = \lim_{n\uparrow\infty} \sum_{j=1}^{n} f(t_{j-1})(B(t_j) - B(t_{j-1})), \qquad t_j = \frac{jT}{n}.$$

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The Itô integral can be used to notationally define differentials and (stochastic) differential equations:

$$X_T := \int_0^T f(t) \mathrm{d}B_t \qquad \Longleftrightarrow \qquad \mathrm{d}X_t = f(t) \mathrm{d}B_t.$$

It is this differential notation that we will mostly exercise moving forward.

Here are some examples of SDE's:

– Let $S_t = \mu t + \sigma B_t$, where B_t is a standard Brownian motion. To determine an SDE, note that by definition:

$\int$ ["L_t"]

$$B_t = \lim_{n\uparrow\infty} B_t = \lim_{n\uparrow\infty} \sum_{j=1}^{n} (B_j - B_{j-1}) = \int_0^{\cancel{T}\ t} 1 \mathrm{d}B_t,$$

$B_j = B(t_j)$

$t_j = \frac{tj}{n}$

i.e.,

$$\mathrm{d}B_t = \mathrm{d}B_t.$$

Combining this with linearity of the Itô integral and our usual understanding of the deterministic differential, we conclude:

$$S_T = \underbrace{\int_0^T \mu \mathrm{d}t}_{} + \underbrace{\int_0^T \sigma \mathrm{d}B_t}_{} \quad\Longrightarrow\quad \mathrm{d}S_t = \mu \mathrm{d}t + \sigma \mathrm{d}B_t.$$

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– From last time:

$$\mathrm{d}(B_t^2) = \mathrm{d}t + 2B_t \mathrm{d}B_t$$

$\int_0^t dB_t = \int_0^t dB(t)$

$x = 1$

$\int_0^1 x^3 dx = \frac{1}{2}\int_0^1 u(x)\, du(x)$

$u = x^2$ $\quad x = 0$

Stochastic processes that are driven by Brownian motion have special terminology:

Suppose X_t is a stochastic process satisfying,

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t + \sigma(X_t, t)\mathrm{d}B_t,$$

where $\mu(\cdot, \cdot)$ and $\sigma(\cdot, \cdot)$ are deterministic functions ("drift" and "volatility", respectively).

Then X_t is called an *Itô process*.

$d(B_t^2) = dt + 2B_t dB_t$

$\mu(t, B_t^2) = 1$

$\sigma(t, B_t^2) = 2\sqrt{B_t^2}$

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Then X_t is called an *Itô process*.

If $\mu = \mu(\cdot)$ and $\sigma = \sigma(\cdot)$, i.e.,

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t,$$

then X_t is an *Itô diffusion*.

Change of variables

A particularly useful tool is a change-of-variable result: Suppose a function satisfies,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu(x;t)$$

What differential equation does $y := f(x,t)$ satisfy?

$$\frac{d}{dt} y = \frac{d}{dt} f(x(t), t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \frac{dx}{dt}$$

$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \cdot \mu(x,t)$$

$$\frac{dy}{dt} = \frac{\partial f}{\partial t} + \mu(x,t) \frac{\partial f}{\partial x}$$

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Through a simple application of the chain rule, we obtain:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} x'(t) \\ &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \mu(u; t) \end{aligned}$$

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If X_t is a trivial Itô process:

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t, \qquad (\sigma \equiv 0)$$

then the standard chain rule would apply for $Y_t = f(X_t, t)$:

$$\mathrm{d}Y_t = \left(\frac{\partial f}{\partial t}(X_t, t) + \mu(X_t, t)\frac{\partial f}{\partial x}(X_t, t)\right)\mathrm{d}t.$$

But this does not apply for $\sigma \neq 0$.

Itô's Lemma

The corresponding chain rule-like result that includes the volatility is the following.

Lemma (Itô's Lemma)

Let X_t be an Itô process:

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t + \sigma(X_t, t)\mathrm{d}B_t.$$

Define $Y_t = f(X_t, t)$. Then Y_t is an Itô process, and satisfies the SDE,

$$\mathrm{d}Y_t = \overbrace{\left(\frac{\partial f}{\partial t}(X_t, t) + \mu(X_t, t)\frac{\partial f}{\partial x}(X_t, t) + \frac{\sigma^2(X_t, t)}{2}\frac{\partial^2 f}{\partial x^2}(X_t, t)\right)}^{\text{drift of } Y_t} \mathrm{d}t$$

$$+ \underbrace{\sigma(X_t, t)\frac{\partial f}{\partial x}(X_t, t)}_{\text{volatility of } Y_t.}\mathrm{d}B_t$$

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More compactly: if we drop the explicit notational dependence on t, X_t, and use f_t, f_x, f_{xx} to denote partial derivatives then:

$$\mathrm{d}Y = \mu_Y \mathrm{d}t + \sigma_Y \mathrm{d}B,$$

with

$$\mu_Y = f_t + \mu f_x + \frac{\sigma^2}{2} f_{xx}, \qquad \sigma_Y = \sigma f_x.$$

Some intuition

Most of Itô's lemma is immediately motivated from deterministic calculus.

$$\frac{dx}{dt} = \mu$$

I.e., from the standard chain rule on deterministic quantities:

$$y(t) = f(x(t), t), \qquad \Longrightarrow \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu(\cancel{y}; t),$$

$x(t)$

we already expect that:

$$\begin{aligned} \mathrm{d}Y_t &= f_t \mathrm{d}t + f_x\left(\mu \mathrm{d}t + \sigma \mathrm{d}B\right) \\ &= \left(f_t + \mu f_x\right)\mathrm{d}t + \sigma f_x \mathrm{d}B. \end{aligned}$$

But in Itô's lemma, there is an additional $\frac{\sigma^2}{2} f_{xx}\mathrm{d}t$ term.

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But in Itô's lemma, there is an additional $\frac{\sigma^2}{2} f_{xx}\mathrm{d}t$ term.

This term arises because $[B]_t = t$, or $(\mathrm{d}B_t)^2 = \mathrm{d}t$:

$$\begin{aligned} f(X(t+\Delta t), t+\Delta t) - f(X(t), t) \approx & f_t \Delta t + (X(t+\Delta t) - X(t)) f_x \\ & + \frac{1}{2}(X(t+\Delta t) - X)^2 f_{xx} + \ldots \end{aligned}$$

The first-order derivatives yield what we already expect. The second order term yields,

$$\frac{1}{2}(X(t+\Delta t) - X)^2 f_{xx} \sim \frac{1}{2}\left(\mu \mathrm{d}t + \sigma \mathrm{d}B_t\right)^2 f_{xx} \sim \frac{1}{2}\sigma^2 f_{xx}(\mathrm{d}B_t)^2 + \ldots = \frac{1}{2}\sigma^2 f_{xx}\mathrm{d}t + \ldots$$

Itô's Lemma is extremely useful in general, but some utility that is particularly useful for us is the ability to identify SDEs for processes.

Example

Recall that

$$\mathrm{d}(B_t^2) = \mathrm{d}t + 2B_t \mathrm{d}B_t$$

Construct an SDE for $e^{(B_t^2)+t}$, and identify the drift and volatility functions.

First show SDE for B_t^2: $dB_t = dB_t$ $(\mu = 0, \ \sigma = 1)$

$X_t = B_t^2 \qquad f(x,t) = x^2$

$X_t = f(B_t, t)$

$[dB_t = \mu\, dt + \sigma\, dB_t]$

Ito's Lemma: $dX_t = \left(\frac{\partial f}{\partial t} + \mu \cdot \frac{\partial f}{\partial x} \right) dt + \frac{\partial f}{\partial x} \sigma \, dB_t$

$+ \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x^2} dt$

$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = 2x, \quad \frac{\partial^2 f}{\partial x^2} = 2$

$\Rightarrow dX_t = (0 + 0 \cdot 2x) dt + 2x \cdot 1 \cdot dB_t$

$+ \frac{1}{2} (1)^2 \cdot 2 \cdot dt \quad (@ \; x = B_t)$

$= 2 B_t \, dB_t + dt$

Given $X_t = B_t^2$ satisfies

$dX_t = \mu \, dt + \sigma \, dB_t$

$\mu = 1, \quad \sigma = 2 B_t,$

Construct an SDE for $Y_t = \exp(B_t^2 + t)$

$= f(B_t, t)$

$f(x, t) = e^{x^2 + t}$

$(dB_t = 0 \, dt + 1 \, dB_t)$

$$dY_t = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right) dt$$

$$+ \sigma \frac{\partial f}{\partial x} dB_t \qquad (\text{evaluated @ } x = B_t)$$

$$\frac{\partial f}{\partial t} = f(x,t) \qquad \frac{\partial f}{\partial x} = 2x\, f(x,t)$$

$$\frac{\partial^2 f}{\partial x^2} = 2f(x,t) + (4x^2) f(x,t)$$

$$\Rightarrow dY_t = \left(f + 2x f \mu + \frac{1}{2} \cdot 4B_t^2 \left[2f + 4x^2 f\right] \right)$$

dt

$$+ 2x f (2B_t)\, dB_t \qquad (@\ x = B_t)$$

$$f(B_t, t) = Y_t$$

$$\Rightarrow dY_t = \left(Y_t + 2B_t Y_t + \frac{1}{2} 4B_t^2 \left(2Y_t + 2B_t^2 Y_t\right] \right) dt$$

$$+ 2B_t Y_t (2B_t)\, dB_t$$

(still to do: write X_t in terms of Y_t, or
write B_t in terms of Y_t)

$$\Rightarrow dY_t = \tilde{\mu}(Y_t, t)\,dt + \tilde{\sigma}(Y_t, t)\,dB_t$$

for some functions $\tilde{\mu}$, $\tilde{\sigma}$.

Geometric Brownian Motion

Back to securities, we assumed availability of continuous-time drift and volatility (μ, σ).

We've seen that a reasonable stochastic model for the log-return $L(t)$ is,

$$L_t = \mu t + \sigma B_t, \quad \Rightarrow \quad dL_t = \mu dt + \sigma dB_t$$

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We then expect that a reasonable model for the security price is,

$$S_t = S_0 e^{L_t} = S_0 e^{\mu t + \sigma B_t}.$$

What kind of SDE does S_t satisfy?

$\Rightarrow dL_t = \mu dt + \sigma dB_t$, $S_t = S_0 e^{L_t} = f(L_t, t)$, $f(x,t) = S_0 e^x$

$\frac{\partial f}{\partial t} = 0.$ $\frac{\partial f}{\partial x} = f$, $\frac{\partial^2 f}{\partial x^2} = f$

Itô: $dS_t = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \left(\frac{\partial f}{\partial x} \sigma \right) dB_t$

$$\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = f(L_t, t) = S_0 e^{L_t} = S_t$$

$$\Rightarrow dS_t = \left(0 + \mu S_t + \frac{1}{2}\sigma^2 S_t\right) dt + (S_t \sigma)\, dB_t$$

$$= \hat{\mu}\, dt + \tilde{\sigma}\, dB_t$$

$$\hat{\mu} = \left(\mu + \frac{1}{2}\sigma^2\right) S_t \qquad \tilde{\sigma} = \sigma S_t$$

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What kind of SDE does S_t satisfy?

Definition

Let μ, σ be constants, $\sigma > 0$. With B_t a standard Brownian motion, suppose S_t is a stochastic process defined by,

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t, \qquad S(t=0) = S_0.$$

Then S_t is a **Geometric Brownian Motion** with drift and volatility (μ, σ).

Note: μ here is slightly different from our CRR μ:

μ_{CRR}, σ_{CRR} $\rightarrow$ $\sigma = \sigma_{CRR}$, $\mu = \mu_{CRR} - \frac{1}{2}\sigma^2_{CRR}$

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Then S_t is a **Geometric Brownian Motion** with drift and volatility (μ, σ).

Note that S_t as defined above corresponds to the process,

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma B_t},$$

which is a $\mathrm{lognormal}\left(\left(\mu - \frac{\sigma^2}{2}\right) t, \sigma^2 t\right)$ random variable.

The gist and summary

We set out to construct a continuous-time analogue of the binomial tree model for securities.

– Geometric Brownian motion, i.e., a stochastic process satisfying an SDE of the form,

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- This model can be viewed as the continuous-time limit of the CRR model for S_n: the price of S_n in the CRR model evolves according to *geometric* increments. Geometric Brownian motion follows a similar principle (as the SDE reveals).

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- This model can be viewed as the continuous-time limit of the CRR model for S_n: the price of S_n in the CRR model evolves according to *geometric* increments. Geometric Brownian motion follows a similar principle (as the SDE reveals).
- One nice thing about the SDE formulation: it's ok if μ or σ vary with time, or even depend on S_t. This model is *flexible*.

How does one numerically simulate SDE paths? It's not quite the same as coin flips. The model is,

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t, \qquad\qquad S(t=0) = S_0.$$

Given an equispaced set of times,

$$t_j = hj, \qquad\qquad h > 0,$$

and for notational ease setting $S_{t_j} = S_j$, then how do we generate a trajectory?

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$$\underbrace{S_{j+1} - S_j}_{dS_t} = \mu S_j \underbrace{(t_{j+1} - t_j)}_{dt = h} + \sigma S_j \overbrace{\underbrace{(B_{j+1} - B_j)}_{dB_t}}^{N(0,h)}.$$

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$$S_{j+1} - S_j = \mu S_j (t_{j+1} - t_j) + \sigma S_j (B_{j+1} - B_j).$$

But $t_{j+1} - t_j = h$, and $B_{j+1} - B_j \sim \mathcal{N}(0, h)$, so this scheme can be written as,

$$S_{j+1} = S_j + \mu h S_j + \sigma S_j \sqrt{h} Z, \qquad Z \sim \mathcal{N}(0,1),$$

with S_0 given and fixed. $= S_j\left(1 + \mu h + \sigma\sqrt{h}\, Z\right)$

This scheme is called the **Euler-Maruyama method**.
(This is exactly how Brownian motion figures were generated on previous slides.)

A digression on SDEs

We won't really use SDE's in more complicated situations than we've covered.

But SDEs are **enormously** useful in various non-finance contexts.

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Around the target is key: of course we can generate completely random things that might not make sense.

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Suppose we tried to generate random images of objects or scenarios: Abstractly, what this means is that there should be a pipeline that accomplishes:

"Mathematics taking over the world" $\rightarrow$ "Drift" μ $\rightarrow$ Generate SDE trajectories

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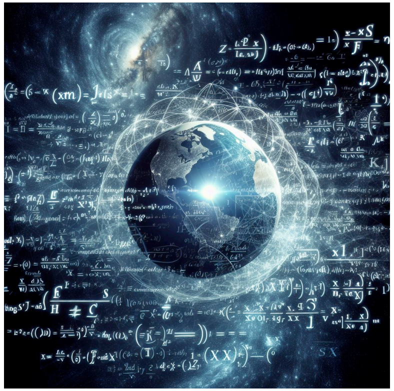
Suppose we tried to generate random images of objects or scenarios: Abstractly, what this means is that there should be a pipeline that accomplishes:

"Mathematics taking over the world" $\longrightarrow$ "Drift" μ $\longrightarrow$ Generate SDE trajectories

That should generate an output like:






Or even more abstractly, the input "target" can be an image itself:

And the output could be SDE-based stochastic diffusions of the input:

This methodology is called creating a **diffusion map**, although there are many variants.

And this is exactly what certain **generative AI** software does, in particular image-based AI generators.

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SDEs are a large part of the mathematics that underlie these deep learning-based models.

The key, very difficult problem is a proper learning of the drift: one has to hit the right target in image space.
(Of course one should also add "noise" in the right ways.)

This is why most diffusion-based generative AI models require lots of data: training the drift and volatility so that the SDE evolution generates meaningful outputs is very hard.

(The figures on the previous slides were generated using Copilot and DALL-E.)

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.