Department of Mathematics, University of Utah

Applied Complex Variables and Asymptotic Methods

MATH 6720 – Section 001 – Spring 2023

Homework 6

Computing Integrals

Due: Tuesday, April 25, 2023

Below, problem C in section A.B is referred to as exercise A.B.C.
Text: *Complex Variables: Introduction and Applications*, Ablowitz & Fokas,

Exercises: 5.7.1
5.7.5
6.1.1
6.1.2
6.2.2
6.2.3
6.2.5
6.3.1
6.3.3

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5.7.1. Show that the "cross ratios associated with the points $(z, 0, 1, -1)$ and $(w, i, 2, 4)$ are $(z+1)/2z$ and $(w-4)(2-i)/2(i-w)$, respectively. Use these to find the bilinear transformation that maps $0, 1, -1$ to $i, 2, 4$.

Solution: The cross-ratio for $(z, 0, 1, -1)$ is given by,

$$X(z, 0, 1, -1) = \frac{(z - -1)(1 - 0)}{(z - 0)(1 - -1)} = \frac{z+1}{2z}.$$

The cross-ratio for $(w, i, 2, 4)$ is given by,

$$X(w, i, 2, 4) = \frac{(w-4)(2-i)}{(w-i)(2-4)} = \left(\frac{i-2}{2}\right)\frac{w-4}{w-i}$$

In order to compute the requested bilinear transformation, we recall that cross-ratios are invariant under such transformations. Therefore, this map is given by,

$$X(z, 0, 1, -1) = X(w, i, 2, 4),$$

which, after some algebra, can be written as,

$$w = w(z) = \frac{(8-3i)z + i}{(3-i)z + 1}$$

5.7.5. Let C_1 be the circle with center $i/2$ passing through 0, and let C_2 be the circle with center $i/4$ passing through 0 (see Figure 5.7.7 in the text). Let D be the region enclosed by C_1

and C_2. Show that the inversion $w_1 = 1/z$ maps D onto the strip $-2 < \mathrm{Im}\,(w)_1 < -1$ and the transformation $w_2 = e^{\pi w_1}$ maps this strip to the upper half plane. Use these results to find a conformal mapping that maps D onto the unit disk.

Solution: We will use the fact that bilinear transformations map generalized circles to generalized circles, and hence also transformed regions bounded by generalized circles to regions bounded by generalized circles. First, we seek to establish that $w_1(z) = 1/z$ transforms D to the strip $-2 < \mathrm{Im}\,(w)_1 < -1$. The sets C_1 and C_2 are given by the set of z satisfying:

$$C_1 : z = \frac{i}{2} + \frac{1}{2}e^{i\theta}, \qquad\qquad C_2 : z = \frac{i}{4} + \frac{1}{4}e^{i\theta},$$

where $\theta \in [0, 2\pi)$ is a free parameter. The image of these points under the map w_1 is given by,

$$C_1 : \quad w_1(z) = \frac{2}{i + e^{i\theta}} = \frac{2(\cos\theta - i - i\sin\theta)}{2(1+\sin\theta)} = \frac{\cos\theta}{1+\sin\theta} - i,$$

and as θ ranges over $[0, 2\pi)$, then the real part of the above expression takes on every extended real value, but $\mathrm{Im}\,(w)_1\,(z) = -1$. Hence

$$w_1(C_1) = \left\{ z \in \mathbb{C} \;\middle|\; \mathrm{Im}\,(z) = -1 \right\}.$$

Similarly,

$$C_2 : \quad w_1(z) = \frac{4}{i + e^{i\theta}} = \frac{2\cos\theta}{1+\sin\theta} - 2i,$$

and hence,

$$w_1(C_2) = \left\{ z \in \mathbb{C} \;\middle|\; \mathrm{Im}\,(z) = -2 \right\}.$$

Finally, we have $w_1(3i/4) = -\frac{4}{3}i$, i.e., $\mathrm{Im}\,(w)_1\,(3i/4) \in (-1, -2)$, which establishes that,

$$w_1(D) = \left\{ z \in \mathbb{C} \;\middle|\; -2 < \mathrm{Im}\,(z) < -1 \right\}.$$

Our next task is to show that the map $w_2 = w_2(z) = e^{\pi z}$ maps the strip $w_1(D)$ to the upper half plane. Note that,

$$-2 < \mathrm{Im}\,(z) < -1 \implies \mathrm{Im}\,(w_2(z)) = \mathrm{Im}\,(e^{\pi z}) = e^{\pi \mathrm{Re}(z)} \sin(\pi \mathrm{Im}\,(z)) > 0$$

where the last inequality uses the fact that $e^x > 0$ for any real x. Thus, we have,

$$w_2(w_1(D)) = \left\{ z \in \mathbb{C} \;\middle|\; \mathrm{Im}\,(z) > 0 \right\}.$$

Note also that w_2 is conformal: $w_2'(z) = \pi e^{\pi z} \neq 0$. To finally map to the unit disk, note that for any $a, b \in \mathbb{C}$ satisfying $\mathrm{Im}\,(a) > 0$ and $|b| = 1$, then the map

$$w_3(z) = b\frac{z-a}{z-\overline{a}},$$

maps the upper half plane to the unit disk. (Again we can check that the real axis, $\mathrm{Im}\,(z) = 0$, is mapped to the unit circle, and $z = a$ in the upper half plane is mapped to the origin.) Thus, for any such choice of a, b, then the composition,

$$w(z) = (w_3 \circ w_2 \circ w_1)(z),$$

maps D to the unit disk, and does so conformally since w_1, w_2, and w_3 are all conformal.

6.1.1.

(a) Consider the function

$$f(\epsilon) = e^{\epsilon}.$$

Find the asymptotic expansion of $f(\epsilon)$ in powers of ϵ as $\epsilon \to 0$.

(b) Similarly for the function

$$f(\epsilon) = e^{-\frac{1}{\epsilon}},$$

find the asymptotic expansion of $f(\epsilon)$ in powers of ϵ as $\epsilon \to 0$.

Solution:

(a) There are a handful of ways to establish this. We'll use Taylor expansions: For ϵ sufficiently small, we have,

$$f(\epsilon) = \sum_{n=0}^{N} \frac{\epsilon^n}{n!} + \mathcal{O}(\epsilon^{N+1}).$$

Hence, we formally have the asymptotic expansion,

$$f(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n, \qquad\qquad a_n = \frac{1}{n!}.$$

(b) For variety, we'll use a different strategy for this part. (Note that we assume $\epsilon > 0$.) If we seek an expansion of the form,

$$f(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n,$$

then the coefficients a_n can be computed recursively by,

$$a_n = \lim_{\epsilon \to 0} \frac{f(\epsilon) - \sum_{m=0}^{n-1} a_m \epsilon^m}{\epsilon^n}, \qquad\qquad n \geq 1.$$

For $n = 0$, a striaghtforward computation shows that,

$$f(\epsilon) = o(\epsilon^0) = o(1),$$

and therefore $a_0 = 0$. Now fix some $n \geq 1$, and suppose that $a_m = 0$ for every $0 \leq m < n$. Then,

$$a_n = \lim_{\epsilon \to 0} \frac{f(\epsilon) - \sum_{m=0}^{n-1} a_m \epsilon^m}{\epsilon^n} = \lim_{\epsilon \to 0} \epsilon^{-n} e^{-\frac{1}{\epsilon}} = 0.$$

Hence, we have shown through induction that $a_n = 0$ for every n, so that we have the asymptotic expansion,

$$f(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n, \qquad\qquad a_n = 0.$$

6.1.2. Show that both the functions $(1+x)^{-1}$ and $(1+e^{-x})(1+x)^{-1}$ possess the same asymptotic expansion as $x \to \infty$.

Solution: We assume x is real and positive. First we compute the asymptotic expansion (in powers of $1/x$) for $(1+x)^{-1}$. We have,

$$\frac{1}{1+x} = \frac{1}{x}\frac{1}{1+\frac{1}{x}} \overset{x \geq 1}{=} \frac{1}{x}\sum_{n=0}^{\infty}\frac{1}{x^n} = \sum_{n=1}^{\infty}\frac{1}{x^n}.$$

Note that asymptotic expansions are additive, so the result will be shown if we can establish that

$$\frac{e^{-x}}{1+x}$$

has an asymptotic expansion of 0. We can accomplish this in a similar fashion to the previous problem: first note that

$$\frac{e^{-x}}{1+x} = o(1), \qquad \text{as } x \to \infty.$$

So that the leading behavior of the asymptotic expansion is 0. Then we again recursively compute asymptotic expansion coefficients,

$$\frac{e^{-x}}{1+x} = \sum_{n=0}^{\infty}\frac{a_n}{x^n} \quad \Longrightarrow \quad a_n = \lim_{x\to\infty}\frac{\frac{e^{-x}}{1+x} - \sum_{m=0}^{n-1}\frac{a_m}{x^m}}{\frac{1}{x^n}},$$

and another inductive argument (see the solution of 6.1.1) shows that all these coefficients vanish. Hence the asymptotic expansion of $\frac{e^{-x}}{1+x}$ is,

$$\frac{e^{-x}}{1+x} = 0, \qquad \text{as } x \to \infty.$$

Then by exercising additivity of asymptotic expansions, we have the identical expansions,

$$\frac{1}{1+x} = \sum_{n=1}^{\infty}\frac{1}{x^n} = \frac{1}{1+x} + \frac{e^{-x}}{1+x} = \frac{1+e^{-x}}{1+x}, \qquad \text{as } x \to \infty.$$

6.2.2. Use integration by parts to obtain the first two terms of the asymptotic expansion of

$$\int_1^{\infty} e^{-k(t^2+1)}\,\mathrm{d}t.$$

Solution: We explicitly exercise integration by parts. (Alternatively, with some preliminary manipulations, one could simply exercise integration by parts formulas from the text.) We have

$$\begin{aligned}\int_1^{\infty} e^{-k(t^2+1)}\,\mathrm{d}t = \int_1^{\infty}\frac{1}{2t}2te^{-k(t^2+1)}\,\mathrm{d}t &\overset{\text{(IbP)}}{=} -\frac{e^{-k(t^2+1)}}{2kt}\Big|_1^{\infty} - \frac{1}{2k}\int_1^{\infty}\frac{1}{t^2}e^{-k(t^2+1)}\,\mathrm{d}t \\ &= \frac{1}{2k}e^{-2k} - \frac{1}{2k}\int_1^{\infty}\frac{1}{t^2}e^{-k(t^2+1)}\,\mathrm{d}t,\end{aligned}$$

and a second integration by parts on the last integral yields,

$$\begin{aligned}\int_1^\infty \frac{1}{t^2} e^{-k(t^2+1)}\,\mathrm{d}t &\overset{\text{(IbP)}}{=} -\frac{1}{2t^3 k} e^{-k(t^2+1)}\Big|_1^\infty - \frac{3}{2k}\int_1^\infty \frac{1}{t^4} e^{-k(t^2+1)}\,\mathrm{d}t \\ &= \frac{e^{-2k}}{2k} - \frac{3}{2k}\int_1^\infty \frac{1}{t^4} e^{-k(t^2+1)}\,\mathrm{d}t,\end{aligned}$$

and so putting things together yields,

$$\int_1^\infty e^{-k(t^2+1)}\,\mathrm{d}t\,\mathrm{d}t = \frac{1}{2k}e^{-2k} - \frac{e^{-2k}}{4k^2} - \frac{3}{4k^2}\int_1^\infty \frac{1}{t^4}e^{-k(t^2+1)}\,\mathrm{d}t,$$

and thus,

$$\int_1^\infty e^{-k(t^2+1)}\,\mathrm{d}t \sim \frac{e^{-2k}}{2k} - \frac{e^{-2k}}{4k^2}$$

6.2.3. Use Watson's Lemma to obtain an infinite asymptotic expansion of

$$I(k) = \int_0^\pi e^{-kt} t^{-\frac{1}{3}} \cos t\,\mathrm{d}t,$$

as $k \to \infty$.

Solution: We translate to the appropriate notation in Watson's Lemma:

- $b = \pi < \infty$
- $f(t) = t^{-1/3}\cos t$, satisfying, $|f(t)| \le \epsilon^{-1/3}$ for all $t \ge \epsilon > 0$.
- Near $t = 0^+$, f behaves like,

  $$f(t) \sim t^\alpha \sum_{n=0}^\infty a_n t^{\beta n},$$

  with $\alpha = -1/3 > -1$, and $\beta = 2 > 0$, and a_n given by,

  $$a_n = \frac{(-1)^n}{(2n)!}$$

Thus, by Watson's Lemma,

$$\begin{aligned}I(k) &\sim \sum_{n=0}^\infty a_n \frac{\Gamma(\alpha+\beta n+1)}{k^{\alpha+\beta n+1}} \\ &= \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}\frac{\Gamma\left(2n+\frac{2}{3}\right)}{k^{2n+2/3}}, \quad k\to\infty\end{aligned}$$

6.2.5. Use Laplace's method to determine the leading behavior (first term) of

$$I(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-k\sin^4 t}\,\mathrm{d}t$$

as $k \to \infty$.

Solution: We have $f(t) = 1$ and $\phi(t) = \sin^4 t$, and thus,

$$\phi'(t) = 4\sin^3 t \cos t,$$

which vanishes only at $t = 0$. Since $\phi(1/2) = \phi(-1/2) > \phi(0)$, then $t = 0$ is the global minimum of ϕ over $[-1/2, 1/2]$. To use Laplace's method, we'll need to compute higher order derivatives:

$$\begin{aligned}
\phi''(t) &= 12\sin^2 t\cos^2 t - 4\sin^4 t = 12\sin^2 t\cos^2 t - 4\phi(t) \\
\phi^{(3)}(t) &= 24\sin t\cos^3 t - 24\sin^3 t\cos t - 4\phi'(t) \\
\phi^{(4)}(t) &= 24\cos^4 t - 72\sin^2 t\cos^2 t - 72\sin^2 t\cos^2 t + 24\sin^4 t - 4\phi''(t).
\end{aligned}$$

We can directly observe that $\phi'(0) = \phi''(0) = \phi^{(3)}(0) = 0$, with $\phi^{(4)}(0) = 24 > 0$. Hence, we cannot use the standard technique of Laplace's method, e.g., Lemma 6.2.3. However, this falls into the category of Example 6.2.10 with $p = 4$, and so equation (6.2.15) can directly be used:

$$I(k) \sim \frac{f(0)e^{k\phi(0)}}{\left(\frac{k\phi^{(4)}(0)}{4!}\right)^{1/4}} \frac{2\Gamma\left(\frac{1}{4}\right)}{4} = \frac{\Gamma\left(\frac{1}{4}\right)}{2k^{1/4}}$$

(Note that for equation 6.2.15, the ϕ function in the book is the negative of the ϕ function here.)

6.3.1. Use integration by parts to obtain the asymptotic expansion as $k \to \infty$ of the following integrals up to order $\frac{1}{k^2}$:
(a) $\int_0^2 (\sin t + t)e^{ikt}\,\mathrm{d}t$
(b) $\int_0^\infty \frac{e^{ikt}}{1+t^2}\,\mathrm{d}t$

Solution:
(a) Via integration by parts, we compute,

$$\begin{aligned}
\int_0^2 (\sin t + t)e^{ikt}\,\mathrm{d}t &\overset{\text{(IbP)}}{=} -i(2+\sin 2)\frac{e^{2ik}}{k} + \frac{i}{k}\int_0^2 (\cos t + 1)e^{ikt}\,\mathrm{d}t \\
&\overset{\text{(IbP)}}{=} -i(2+\sin 2)\frac{e^{2ik}}{k} + \frac{1}{k^2}\left[e^{2ik}(1+\cos 2) - 2\right] + \frac{1}{k^2}\int_0^2 \sin t\; e^{ikt}\,\mathrm{d}t.
\end{aligned}$$

Thus, to order $1/k^2$,

$$\int_0^2 (\sin t + t)e^{ikt}\,\mathrm{d}t \sim -i(2+\sin 2)\frac{e^{2ik}}{k} + \frac{1}{k^2}\left[e^{2ik}(1+\cos 2) - 2\right], \quad k \to \infty.$$

(b) Again via integration by parts:

$$\begin{aligned}
\int_0^\infty \frac{e^{ikt}}{1+t^2}\,\mathrm{d}t &\overset{\text{(IbP)}}{=} \frac{i}{k} + \frac{2}{ik}\int_0^\infty \frac{te^{ikt}}{(1+t^2)^2}\,\mathrm{d}t \\
&\overset{\text{(IbP)}}{=} \frac{i}{k} + \frac{2}{ik}\left[0 - \frac{1}{ik}\int_0^\infty \frac{(1-3t^2)e^{ikt}}{(1+t^2)^3}\,\mathrm{d}t\right],
\end{aligned}$$

since the boundary terms for the second integration by parts vanish. Thus to order $1/k^2$,

$$\int_0^\infty \frac{e^{ikt}}{1+t^2}\,\mathrm{d}t \sim \frac{i}{k}, \quad k \to \infty$$

6.3.3. Use the method of stationary phase to find the leading behavior of the following integrals as $k \to \infty$:

(a) $\int_0^1 \tan t \, e^{ikt^4} \, \mathrm{d}t$

(b) $\int_{\frac{1}{2}}^{2} (1+t) e^{ik\left(\frac{t^3}{3} - t\right)} \, \mathrm{d}t$

Solution:

(a) As with many of these problems, there is more than one way to proceed. For example, for this problem one can directly use equation (6.3.15b) in the text. We will proceed in a different manner (obtaining the same result). First we note that $\phi(t) = t^4$, and so $\phi'(t) = 0$ only when $t = 0$. Hence, it is the neighborhood of $t = 0$ that produces the dominant contribution to the integral. In a neighbhorhood of $t = 0$, we have,

$$\tan t \sim t,$$

and hence we have

$$\int_0^1 \tan t \, e^{ikt^4} \, \mathrm{d}t \sim \int_0^1 t e^{ikt^4} \, \mathrm{d}t,$$

and since integrating over all $t > 0$ does not introduce any terms comparable to the integration around $t = 0$, we in turn have,

$$\int_0^1 \tan t \, e^{ikt^4} \, \mathrm{d}t \sim \int_0^1 t e^{ikt^4} \, \mathrm{d}t \sim \int_0^\infty t e^{ikt^4} \, \mathrm{d}t$$

Owing to equation (6.3.5) in the text with $\gamma = 1$, $\nu = k$, and $p = 4$, then

$$\int_0^\infty t e^{ikt^4} \, \mathrm{d}t = \frac{1}{\sqrt{k}} \frac{\Gamma\left(\frac{1}{2}\right)}{4} e^{i\pi/4} = \frac{1}{\sqrt{k}} \frac{\sqrt{\pi} e^{i\pi/4}}{4}.$$

Putting all this together, we have,

$$\int_0^1 \tan t \, e^{ikt^4} \, \mathrm{d}t \sim \int_0^\infty t e^{ikt^4} \, \mathrm{d}t \sim \frac{1}{\sqrt{k}} \frac{\sqrt{\pi} e^{i\pi/4}}{4}$$

Note that the above approach is *not* different from the approach used to derive equation (6.3.15b) in the text. Indeed, this equation is derived by more formally performing the steps above.

(b) For this integral, we have $\phi(t) = \frac{t^3}{3} - t$, which satisfies,

$$\phi'(1) = 0, \qquad \phi'(t) \neq 0, \quad \text{for } t \in \left[\frac{1}{2}, 2\right] \setminus \{1\}.$$

Note that while $f(t) \neq 0$ at the endpoints, this only affects higher order contributions and not the leading order behavior. (I.e., we can still use stationary phase to determine the leading order behavior.) Then we have that,

$$\begin{aligned} \phi(t) - \phi(1) &\sim \frac{\phi''(1)}{2}(t-1)^2 + o((t-1)^2) = (t-1)^2 + o((t-1)^2), \\ f(t) &\sim 2(t-1)^0 + (t-1) = 2 + o(1), \end{aligned}$$

and hence we can use equation (6.3.10) in the text with $c = 1$, $\alpha = 1$, $\beta = 2$, $\gamma = 0$, $\mu = 1$, $\phi(1) = -2/3$, to obtain,

$$\int_{\frac{1}{2}}^{2} (1+t) e^{ik\left(\frac{t^3}{3} - t\right)} \, \mathrm{d}t \sim e^{-2ik/3} 2\Gamma\left(\frac{1}{2}\right) e^{i\pi/4} \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{k}} \left(2\sqrt{\pi} e^{-2ik/3} e^{i\pi/4}\right)$$