Department of Mathematics, University of Utah
Applied Complex Variables and Asymptotic Methods
MATH 6720 – Section 001 – Spring 2023
Homework 5
Computing Integrals

Due: Friday, April 7, 2023

Below, problem C in section A.B is referred to as exercise A.B.C.
Text: *Complex Variables: Introduction and Applications*, Ablowitz & Fokas,

Exercises: 4.3.2
4.3.3
4.3.7, part (a) only. Note that $0 < k < 1$ is the correct restriction on k.
4.3.13, only compute the first integral, i.e., the one involving $x^{1/2}\log x$.
In addition, for this section the text considers the principal branch of $\log z$ and $z^{1/2}$ to correspond to $z = re^{i\theta}$ for $\theta \in [0, 2\pi)$.

Submit your homework assignment on Canvas via Gradescope.

4.3.2. Show that,

$$\int_0^\infty \frac{\sin x}{x(x^2+1)}\,\mathrm{d}x = \frac{\pi}{2}\left(1 - \frac{1}{e}\right).$$

Solution: We start by defining,

$$f(z) = \frac{e^{iz}}{z(z^2+1)}.$$

Then we have,

$$J := \int_0^\infty f(z)\,\mathrm{d}z, \qquad\qquad I = \mathrm{Im}\,(J)\,,$$

where I is the integral we seek to compute. We will evaluate J using the Cauchy Residue Theorem, with a closed loop consisting of (i) a radius R semicircular contour C_R centered at 0 in the upper half-place with large R, (ii) the integral along the real interval $I_- = (-R, -\epsilon)$ for $\epsilon > 0$ small, (iii) the semicirular contour C_ϵ in the upper half plane centered at 0, (iv) the integral along the real interval $I_+ = (\epsilon, R)$. We will take limits as $R \uparrow \infty$ and $\epsilon \downarrow 0$. We proceed to compute these integrals.
First, we have that,

$$\lim_{\epsilon\to 0}\int_{C_\epsilon} f(z)\,\mathrm{d}z = -i\pi\mathrm{Res}(f;0) = -i\pi\frac{e^0}{(0^2+1)} = -i\pi.$$

since C_ϵ sweeps out an angle of π with clockwise orientation. For $|z| = R > 1$, we have,

$$\left|\frac{1}{z(z^2+1)}\right| \leq \frac{1}{R(R^2-1)} \overset{R\to\infty}{\to} 0,$$

and hence by Jordan's Lemma,

$$\lim_{R\to\infty}\int_{C_R} f(z)\,\mathrm{d}z = \lim_{R\to\infty}\int_{C_R} \frac{e^{iz}}{z(z^2+1)}\,\mathrm{d}z = 0.$$

We next compute the two integrals on the real line:

$$\lim_{\epsilon\to 0^+, R\to\infty}\int_{I_+} f(z)\,\mathrm{d}z = J,$$

and

$$\lim_{\epsilon\to 0^+, R\to\infty}\int_{I_-} f(z)\,\mathrm{d}z = \lim_{\epsilon\to 0^+, R\to\infty} -\int_{\epsilon}^{\infty} \frac{e^{-ix}}{x(x^2+1)}\,\mathrm{d}x = -\overline{J}.$$

Finally, the only residue of f in the upper half plane is located at $z = i$:

$$2\pi i \mathrm{Res}(f; i) = 2\pi i \frac{e^{-1}}{i(2i)} = -\frac{i\pi}{e}$$

Finally, the Cauchy Residue Theorem yields:

$$\lim_{\epsilon\to 0^+, R\to\infty}\left[\int_{C_R} f(z)\,\mathrm{d}z + \int_{I_-} f(z)\,\mathrm{d}z + \int_{C_\epsilon} f(z)\,\mathrm{d}z + \int_{I_+} f(z)\,\mathrm{d}z\right] = 2\pi i \mathrm{Res}(f; i),$$

i.e.,

$$-i\pi + J - \overline{J} = -\frac{i\pi}{e} \quad\Longrightarrow\quad I = \mathrm{Im}\,(J) = \frac{1}{2}\left(\pi - \frac{\pi}{e}\right) = \frac{\pi}{2}\left(1 - \frac{1}{e}\right)$$

4.3.3. Show that,

$$\int_{-\infty}^{\infty} \frac{\cos x - 1}{x^2(x^2+a^2)}\,\mathrm{d}x = -\frac{\pi}{a^2} + \frac{\pi}{a^3}\left(1 - e^{-a}\right), \quad a > 0.$$

Solution: We use the same contour as in the previous problem's solution (4.3.2), in particular with the curves $I_\pm$, C_ϵ, and C_R. We define,

$$f(z) = \frac{e^{iz} - 1}{z^2(z^2+a^2)},$$

whose single residue in the upper half-plane is at $z = ia$:

$$2\pi i \mathrm{Res}(f; ia) = 2\pi i \frac{e^{-a} - 1}{-a^2(2ia)} = \pi \frac{1 - e^{-a}}{a^3}.$$

To evaluate along C_R, we note that for $|z| = R$,

$$\left|\frac{1}{z^2 + (z^2+a^2)}\right| \leq \frac{1}{R^2(R^2 - a^2)} \overset{R\to\infty}{\to} 0,$$

and hence by a combination of Jordan's Lemma, and the result that the integral along C_R of a rational function $P(z)/Q(z)$ with $\deg Q(z) \geq \deg P(z) + 2$ goes to 0, we have,

$$\lim_{R\to\infty} \int_{C_R} f(z)\,\mathrm{d}z = 0.$$

To evaluate along C_ϵ, first note that,

$$f(z) = \frac{e^{iz} - 1}{z^2(z^2 + a^2)} = \frac{1}{z^2 + a^2}\left(\frac{i}{z} - \frac{1}{2} + \dots\right),$$

and hence f as a simple pole at $z = 0$ with $\mathrm{Res}(f; 0) = i/a^2$. Then,

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} f(z)\,\mathrm{d}z = -i\pi \mathrm{Res}(f; 0) = \pi/a^2.$$

On the intervals $I_\pm$ we have, after taking limits:

$$\int_{I_+} f(z)\,\mathrm{d}z \to \int_0^\infty \frac{e^{ix} - 1}{x^2(x^2 + a^2)}\,\mathrm{d}x$$

$$\int_{I_-} f(z)\,\mathrm{d}z \to \int_{-\infty}^0 \frac{e^{ix} - 1}{x^2(x^2 + a^2)}\,\mathrm{d}x = \int_0^\infty \frac{e^{-ix} - 1}{x^2(x^2 + a^2)}\,\mathrm{d}x$$

Finally, putting things together with the Cauchy Residue Theorem yields,

$$\int_0^\infty 2\frac{\cos x - 1}{x^2(x^2 + a^2)}\,\mathrm{d}x + \frac{\pi}{a^2} = \pi\frac{1 - e^{-a}}{a^3},$$

and using the fact that the integrand above is even, this implies

$$\int_{-\infty}^\infty \frac{\cos x - 1}{x^2(x^2 + a^2)}\,\mathrm{d}x = -\frac{\pi}{a^2} + \frac{\pi}{a^3}\left(1 - e^{-a}\right)$$

4.3.7. Use the keyhole contour of Figure 4.3.6 in the text to show that on the principal branch of x^k,

(a)

$$I(a) = \int_0^\infty \frac{x^{k-1}}{(x + a)}\,\mathrm{d}x = \frac{\pi}{\sin k\pi} a^{k-1}, \quad 0 < k < 1, \quad a > 0$$

Solution: We use the same notation as in the figure: C_ϵ denotes a circle of radius $\epsilon > 0$ traversed clockwise with a small opening at $\arg z = 0$, and C_R denotes a circle of radius $R \gg 1$ with a small opening at $\arg z = 0$ traversed counterclockwise. We let I_+ denote the integral along $[\epsilon, R]$ with small positive imaginary part, and I_- the same integral but small negative imaginary part. Define

$$f(z) = \frac{z^{k-1}}{z + a}.$$

We begin by computing the (single) residue inside the contour at $z = -a$:

$$2\pi i \mathrm{Res}(f; -a) = 2\pi i(ae^{i\pi})^{k-1} = -2\pi i a^{k-1} e^{i\pi k}.$$

On the contour C_R with $|z| = R$, we have,

$$|zf(z)| \leq \frac{RR^{k-1}}{R-a} = R^{k-1}\frac{1}{1-a/R} \overset{R\to\infty}{\to} 0,$$

where we have used $k-1 \in (-1,0)$ since $0 < k < 1$. Since this limit is uniform in z, then

$$\lim_{R\to\infty} \int_{C_R} f(z)\,\mathrm{d}z = 0.$$

A similar computation can be carried out on C_ϵ, where $|z| = \epsilon$ and $\epsilon \ll 1$:

$$|zf(z)| \leq \frac{\epsilon^k}{a-\epsilon} \overset{\epsilon\to 0^+}{\to} 0,$$

which holds uniformly in z where again we have used $0 < k < 1$. To understand the integrals on $I_\pm$, we define the branch of the function z^k to be so that $\arg z \in [0, 2\pi)$. The integral along I_+ is given via the parameterization $z = x$,

$$\int_\epsilon^R \frac{x^{k-1}}{x-a}\,\mathrm{d}x \to I(a),$$

and along I_- we use the parameterization $z = xe^{2\pi i}$ to yield[1],

$$\int_R^\epsilon \frac{x^{k-1}e^{2\pi i(k-1)}}{xe^{2\pi i}-a} e^{2\pi i}\,\mathrm{d}x = -e^{2\pi ik}\int_\epsilon^R \frac{x^{k-1}}{x+a}\,\mathrm{d}x \to -e^{2\pi ik} I(a).$$

Putting everything together with the Cauchy Residue Theorem yields,

$$I(a) = -2\pi i a^{k-1} e^{i\pi k}\frac{1}{1-e^{2\pi ik}} = \pi a^{k-1}\frac{2i}{e^{i\pi k}-e^{-i\pi k}} = \frac{\pi}{\sin k\pi} a^{k-1}$$

4.3.13. Use the keyhole contour of Figure 4.3.6 to show for the principal branch of $x^{1/2}$ and $\log x$,

$$\int_0^\infty \frac{x^{1/2}\log x}{(1+x^2)}\,\mathrm{d}x = \frac{\pi^2}{2\sqrt{2}}$$

Solution: We use the same notation for the keyhole contour as in the solution to the previous problem (4.3.7), in particular for the contours C_R, C_ϵ, and $I_\pm$. Define,

$$f(z) = \frac{z^{1/2}\log z}{1+z^2},$$

where for both $z^{1/2}$ and $\log z$ we define our branch as that associated to $\arg z \in [0, 2\pi)$. This function has two residues inside the keyhole contour located at $z = \pm i$:

$$2\pi i \mathrm{Res}(f; i) = 2\pi i \frac{i^{1/2}\log i}{2i} = 2\pi i\frac{e^{i\pi/4} i\frac{\pi}{2}}{2i} = i\frac{\pi^2}{2}e^{i\pi/4}$$
$$2\pi i \mathrm{Res}(f; -i) = 2\pi i \frac{(-i)^{1/2}\log(-i)}{-2i} = 2\pi i\frac{e^{i3\pi/4} i\frac{3\pi}{2}}{-2i} = -i\frac{3\pi^2}{2}e^{i3\pi/4},$$

[1] Techically, the parameterization is $z = xe^{i(2\pi-\delta)}$ for infinitesimal $\delta > 0$.

so that,

$$2\pi i\left(\mathrm{Res}(f;i)+\mathrm{Res}(f;-i)\right)=\frac{\pi^2}{2}e^{i\pi/4}(3+i)=\pi^2\left(\frac{1}{\sqrt{2}}+i\sqrt{2}\right).$$

On the contour C_R, we note that for $|z|=R$ with $R>1$:

$$|zf(z)|=\frac{|z|^{3/2}|\log z|}{|z^2+1|}\leq\frac{R^{3/2}\left(\log R+2\pi\right)}{R^2-1}\overset{R\uparrow\infty}{\to}0,$$

uniformly for $z\in C_R$, which implies,

$$\lim_{R\to\infty}\int_{C_R}f(z)\,\mathrm{d}z=0.$$

Similarly on C_ϵ, for $|z|=\epsilon$ and $\epsilon<1$, we have,

$$|zf(z)|=\frac{|z|^{3/2}|\log z|}{|z^2+1|}\leq\frac{\epsilon^{3/2}\left(\log\epsilon+2\pi\right)}{1-\epsilon^2}\overset{\epsilon\downarrow 0}{\to}0,$$

again uniformly for $z\in C_\epsilon$, and therefore,

$$\lim_{\epsilon\to\infty}\int_{C_\epsilon}f(z)\,\mathrm{d}z=0.$$

We can now compute the integrals on the contours $I_\pm$. On I_+, we use the parameterization $z=x$ with x real to obtain,

$$\int_{I_+}f(z)\,\mathrm{d}z=\int_\epsilon^R\frac{x^{1/2}\log x}{x^2+1}\,\mathrm{d}x\xrightarrow{R\uparrow\infty,\epsilon\downarrow 0}\int_0^\infty\frac{x^{1/2}\log x}{x^2+1}\,\mathrm{d}x=:J.$$

On I_-, we use the parameterization $z=xe^{2\pi i}$ to yield,

$$\begin{aligned}\int_{I_-}f(z)\,\mathrm{d}z&=\int_R^\epsilon\frac{(xe^{2\pi i})^{1/2}\log xe^{2\pi i}}{1+(xe^{2\pi i})^2}e^{2\pi i}\,\mathrm{d}x\\&=\int_\epsilon^R\frac{x^{1/2}\left(\log x+2\pi i\right)}{1+x^2}\,\mathrm{d}x\\&\xrightarrow{R\uparrow\infty,\epsilon\downarrow 0}J+2\pi i\int_0^\infty\frac{x^{1/2}}{1+x^2}\,\mathrm{d}x.\end{aligned}$$

Combining all this with the Cauchy Residue Theorem (and taking limits) yields,

$$2J+2\pi i\int_0^\infty\frac{x^{1/2}}{x^2+1}\,\mathrm{d}x=\frac{\pi^2}{\sqrt{2}}+i\sqrt{2}\pi^2,$$

and taking real parts of the above equality implies,

$$\int_0^\infty\frac{x^{1/2}\log x}{(1+x^2)}\,\mathrm{d}x=J=\frac{\pi^2}{2\sqrt{2}}$$