Department of Mathematics, University of Utah

Applied Complex Variables and Asymptotic Methods

MATH 6720 – Section 001 – Spring 2023

Homework 4

Residue Calculus

Due: Monday, March 27, 2023

Below, problem C in section A.B is referred to as exercise A.B.C.

Text: *Complex Variables: Introduction and Applications*, Ablowitz & Fokas,

Exercises: 4.1.1, parts (b), (d), and (e)
4.1.2, parts (b) and (c)
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4.2.1, parts (b) and (d)
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Submit your homework assignment on Canvas via Gradescope.

4.1.1. Evaluate the integrals $\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z$, where C is the unit circle centered at the origin and $f(z)$ is given below.

(b) $\frac{\cosh(1/z)}{z}$

(d) $\frac{\log(z+2)}{2z+1}$, principal branch

(e) $\frac{z+1/z}{z(2z-1/2z)}$

Solution: We will evaluate these integrals using the Cauchy Residue Theorem.

(b) The only singularity inside C is at $z = 0$, which is an essential singularity due to the Laurent expansion of $\cosh(1/z)$. Therefore, we compute the residue using the Laurent expansion of f around 0:

$$f(z) = \frac{1}{z}\left(\sum_{n=0}^{\infty} \frac{1}{z^{2n}(2n)!}\right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z^{2n+1}(2n)!},$$

and therefore $\mathrm{Res}(f;0) = 1$, so the Residue Theorem implies that

$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \mathrm{Res}(f;0) = 1.$$

(d) For the principal branch of $w \mapsto \log w$, we take $w = re^{i\theta}$ with $\theta \in [-\pi, \pi)$. The singularity of the numerator $\log(z+2)$ lies at $z = -2$, which is outside C, and the denominator has a simple zero at $z = -1/2$, which lies inside C. Therefore, we need only compute $\mathrm{Res}(f;-1/2)$. This can be directly computed as,

$$\mathrm{Res}(f;-1/2) = \frac{\log(-1/2+2)}{(2z+1)'|_{z=-1/2}} = \frac{1}{2}\log\frac{3}{2}.$$

By the Residue Theorem,

$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \mathrm{Res}(f;-1/2) = \frac{1}{2}\log\frac{3}{2}.$$

(e) We rewrite this function as,

$$f(z) = \frac{2z^2+2}{z(4z^2-1)}.$$

Since the denominator is a polynomial with simple zeros, then f has simple poles at $z=0$ and $z=\pm 1/2$, all of which lie inside C. We compute the corresponding residues as follows:

$$\begin{aligned}
\mathrm{Res}(f;0) &= \frac{(2z^2+2)\big|_{z=0}}{(z(4z^2-1))'\big|_{z=0}} = \frac{2}{-1} = -2,\\
\mathrm{Res}(f;+1/2) &= \frac{(2z^2+2)\big|_{z=1/2}}{(z(4z^2-1))'\big|_{z=1/2}} = \frac{5/2}{z(8z)\big|_{z=1/2}} = \frac{5}{4},\\
\mathrm{Res}(f;-1/2) &= \frac{(2z^2+2)\big|_{z=-1/2}}{(z(4z^2-1))'\big|_{z=-1/2}} = \frac{5/2}{z(8z)\big|_{z=-1/2}} = \frac{5}{4}.
\end{aligned}$$

Then by the Residue Theorem,

$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \mathrm{Res}(f;0) + \mathrm{Res}(f;1/2) + \mathrm{Res}(f;-1/2) = \frac{1}{2}.$$

4.1.2. Evaluate the integrals $\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z$, where C is the unit circle centered at the origin with $f(z)$ given below. Do these problems by both (i) enclosing the singular points inside C and (ii) enclosing the singular points outside C (by including the point at infinity). Show that you obtain the same result in both cases.

(b) $\frac{z^2+1}{z^3}$
(c) $z^2 e^{-1/z}$

Solution: We will evaluate these integrals using the Cauchy Residue Theorem using two different ways. First we note that as a consequence of the definition,

$$\mathrm{Res}(f;\infty) := \lim_{R\to\infty}\frac{1}{2\pi i}\oint_{\partial B_R(0)} f(z)\,\mathrm{d}z,$$

then $\sum_{j=1}^{M}\mathrm{Res}(f;z_j) = \mathrm{Res}(f;\infty)$, where $\{z_j\}_{j=1}^{M}$ are the singularities of f in the finite plane $\mathbb{C}$.

(b) The only singularity inside C is at $z=0$, and there are no singularities in the finite plane outside C. Therefore,

$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \mathrm{Res}(f;0) = \mathrm{Res}(f;\infty).$$

The Laurent expansion of f at 0 is given by the function itself, $f(z) = \frac{1}{z} + \frac{1}{z^3}$, so $\mathrm{Res}(f;0) = 1$. To compute the Residue at infinity, we use the formula,

$$\mathrm{Res}(f(z);\infty) = \mathrm{Res}\left(\frac{1}{w^2}f\left(\frac{1}{w}\right);0\right) = \mathrm{Res}\left(\frac{1}{w}+w;0\right) = 1.$$

Hence we have,

$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \mathrm{Res}(f;\infty) = 1,$$
$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \mathrm{Res}(f;0) = 1,$$

as expected.

(c) Again, the only singularity inside C is at $z = 0$, and there are no singularities in the finite plane outside C. So again we have,

$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \mathrm{Res}(f;0) = \mathrm{Res}(f;\infty).$$

The point $z = 0$ is an essential singularity for f, so we compute the Laurent expansion:

$$f(z) = z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n n!} = z^2 - z + \frac{1}{2} - \frac{1}{6z} + \ldots,$$

so $\mathrm{Res}(f;0) = -1/6$. The residue at infinity is given by,

$$\mathrm{Res}(f(z);0) = \mathrm{Res}\left(\frac{1}{w^2} f\left(\frac{1}{w}\right);0\right) = \mathrm{Res}\left(\frac{e^{-w}}{w^4};0\right) = -\frac{1}{6}.$$

Therefore, we have,

$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \mathrm{Res}(f;\infty) = -\frac{1}{6},$$
$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z = \mathrm{Res}(f;0) = -\frac{1}{6}.$$

4.1.8. Suppose $f(z)$ is a meromorphic function (i.e., $f(z)$ is analytic everywhere in the finite z plane except at isolated points where it has poles) with N simple zeros (i.e., $f(z_0) = 0$, $f'(z_0) \neq 0$) and M simple poles inside a circle C. Show

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = N - M.$$

Solution: Note that $\frac{f'(z)}{f(z)}$ has singularities only where either $f(z)$ and/or $f'(z)$ have singularities, or where $f(z)$ has zeros. Since $f'(z)$ is analytic inside C everywhere that f is analytic, then the only singularities of $\frac{f'(z)}{f(z)}$ occur where f has singularities or zeros. Let $\{z_j\}_{j=1}^N$ be the zeros (simple) of f, and let $\{w_k\}_{k=1}^M$ be the singularities (simple poles) of f. Then by the Residue Theorem,

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{j=1}^{N} \mathrm{Res}\left(\frac{f'(z)}{f(z)};z_j\right) + \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f'(z)}{f(z)};w_k\right). \tag{1a}$$

For a fixed j, since z_j is a simple zero, then in a neighborhood of z_j we have,

$$f(z) = (z - z_j)g(z), \quad g(z) \neq 0,\ g(z) \text{ is analytic.}$$

In this neighborhood, we compute,

$$f'(z) = g(z) + (z - z_j)g'(z) \quad \Longrightarrow \quad \frac{f'(z)}{f(z)} = \frac{1}{z - z_j} + \frac{g'(z)}{g(z)},$$

with $g'(z)/g(z)$ analytic in this neighborhood since g has no poles or singularities in this neighborhood. Therefore,

$$\text{Res}\left(\frac{f'(z)}{f(z)}; z_j\right) = 1. \tag{1b}$$

Now consider for a fixed k a similar computation in a neighborhood of w_k, which is a simple pole:

$$f(z) = \frac{h(z)}{z - w_k}, \quad h(z) \neq 0, \;\; h(z) \text{ is analytic.}$$

Then in this neighborhood, we have

$$f'(z) = -\frac{h(z)}{(z - w_k)^2} + \frac{h'(z)}{z - w_k} \quad \Longrightarrow \quad \frac{f'(z)}{f(z)} = -\frac{1}{z - w_k} + \frac{h'(z)}{h(z)},$$

and $h'(z)/h(z)$ is analytic in this neighborhod since h has no zeros or singularities in this neighborhood. Therefore,

$$\text{Res}\left(\frac{f'(z)}{f(z)}; w_k\right) = -1. \tag{1c}$$

Combining the three equalities (1a), (1b), and (1c) proves the desired result.

4.2.1. Evaluate the following real integrals.
(b) $\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}$, $a^2 > 0$
(d) $\int_0^\infty \frac{\mathrm{d}x}{x^6+1}$

Solution: The main technique will be using the Cauchy Residue Theorem on a closed contour that is the union of the real interval $[-R, R]$ with a circular contour C_R, with C_R defined as the portion of $\partial B_R(0)$ in the upper half-plane. I.e., for a suitably defined $f(z)$ with singularities $\{z_j\}_{j=1}^M$ in the upper half-plane, we will compute via the Cauchy Residue Theorem,

$$\lim_{R\uparrow\infty} \left[\int_{-R}^{R} f(z)\,\mathrm{d}z + \int_{C_R} f(z)\,\mathrm{d}z\right] = 2\pi i \sum_{j=1}^{M} \text{Res}(f; z_j).$$

In this case, we will have,

$$\lim_{R\uparrow\infty} \int_{C_R} f(z)\,\mathrm{d}z = 0 \quad \Longrightarrow \quad \text{PV} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_{j=1}^{M} \text{Res}(f; z_j), \tag{2}$$

and the equality above will be our main strategy for computing these integrals.

(b) We assume without loss that $a > 0$ (since $a \leftarrow -a$ leaves the integral unchanged). Since the integrand is even, then

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2} = \frac{1}{2}\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x, \quad \text{where } f(x) \coloneqq \frac{1}{x^2+a^2}$$

(The principal value is not needed here, but we'll continue to use it.) With C_R the circular contour described above, we have

$$\lim_{R\uparrow\infty} \int_{C_R} f(z)\,\mathrm{d}x = 0,$$

since f is a rational function of z with $f(z) = P(z)/Q(z)$ and $\deg Q \geq \deg P + 2$. There is a lone singularity of f in the upper half-plane at $z = ia$, which is a pole of order 2 with residue,

$$2\pi i \mathrm{Res}(f; ia) = 2\pi i \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}z}\left((z-ia)^2 f(z)\right)\big|_{z=ia} = 2\pi i \frac{-2}{(ia+ia)^3} = \frac{\pi}{2a^3}.$$

Using these in (2) yields,

$$\mathrm{PV}\int_{-\infty}^{\infty} f(z)\,\mathrm{d}z = \frac{\pi}{2a^3},$$

and therefore,

$$\int_0^\infty \frac{1}{(x^2+a^2)^2}\,\mathrm{d}x = \frac{\pi}{4a^3}, \quad a > 0,$$

and thus for arbitrary real $a \neq 0$, we have

$$\int_0^\infty \frac{1}{(x^2+a^2)^2}\,\mathrm{d}x = \frac{\pi}{4|a|^3}.$$

(d) Since the integrand is even, then

$$\int_0^\infty \frac{\mathrm{d}x}{x^6+1} = \frac{1}{2}\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x, \quad \text{where } f(x) \coloneqq \frac{1}{x^6+1}$$

(Again, the principal value is not really needed here.) With C_R the circular contour described above, we have

$$\lim_{R\uparrow\infty} \int_{C_R} f(z)\,\mathrm{d}x = 0,$$

since f is a rational function of z with $f(z) = P(z)/Q(z)$ and $\deg Q \geq \deg P + 2$. The function f has 6 simple poles in $\mathbb{C}$, and three of them are in the upper half-plane. These are located at:

$$z_1 = e^{i\pi/6}, \qquad z_2 = e^{i\pi/2}, \qquad z_3 = e^{i5\pi/6}.$$

The residues at these points are given by,

$$
\begin{aligned}
2\pi i \operatorname{Res}(f; z_1) &= \frac{2\pi i}{6z_1^5} = \frac{\pi}{3} e^{-\pi i/3}, \\
2\pi i \operatorname{Res}(f; z_2) &= \frac{2\pi i}{6z_2^5} = \frac{\pi}{3}, \\
2\pi i \operatorname{Res}(f; z_3) &= \frac{2\pi i}{6z_3^5} = \frac{\pi}{3} e^{\pi i/3}
\end{aligned}
$$

Therefore, by (2):

$$
\mathrm{PV} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_{j=1}^{3} \operatorname{Res}(f; z_j) = \frac{\pi}{3}\left(1 + 2\cos\frac{\pi}{3}\right) = \frac{2\pi}{3},
$$

and therefore,

$$
\int_0^\infty \frac{1}{x^6+1}\,\mathrm{d}x = \frac{\pi}{3}
$$

4.2.2. Evaluate the following real integrals by residue integration:
(a) $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2}\,\mathrm{d}x, \quad a^2 > 0$
(d) $\int_0^\infty \frac{\cos kx}{x^4+1}\,\mathrm{d}x, \quad k$ real
(g) $\int_0^{\pi/2} \sin^4\theta\,\mathrm{d}\theta$

Solution:
(a) Define I as the integral we seek to compute. Then

$$
I = \operatorname{Im}(J), \qquad\qquad J := \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+a^2}\,\mathrm{d}x,
$$

and we will compute J to determine I. Define $f(z)$ as the rational part of the integrand for J:

$$
f(z) := \frac{z}{z^2+a^2}.
$$

Let C_R be the circular contour that is the portion of $\partial B_R(0)$ in the upper half-plane. Then $[-R, R]$ unioned with C_R is a closed contour. By the Cauchy Residue Theorem,

$$
\lim_{R\uparrow\infty} \int_{C_R} f(z) e^{iz}\,\mathrm{d}z + \mathrm{PV}\int_{-\infty}^{\infty} e^{iz} f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{M} \operatorname{Res}(f(z)e^{iz}; z_j), \tag{3}
$$

where $\{z_j\}_{j=1}^M$ the singularities of f in the upper half-plane. The function f has one such lone singularity ($M = 1$) at $z = i|a|$, with residue,

$$
2\pi i \operatorname{Res}(f(z)e^{iz}; i|a|) = 2\pi i \frac{i|a| e^{-|a|}}{2i|a|} = e^{-|a|}\pi i.
$$

Note that

$$
\lim_{R\uparrow\infty} \max_{z\in C_R} |f(z)| \le \lim_{R\uparrow\infty} \frac{1}{R} = 0,
$$

and so f uniformly decays to 0 on C_R as $R \uparrow \infty$. Thus, by Jordan's Lemma,

$$\lim_{R\uparrow\infty} \int_{C_R} f(z) e^{iz} \, \mathrm{d}z = 0.$$

Putting all this together in (3), we have,

$$J = \mathrm{PV} \int_{-\infty}^{\infty} e^{iz} f(z) \, \mathrm{d}z = i\pi e^{-|a|},$$

and therefore,

$$I = \mathrm{Im}\,(J) = \pi e^{-|a|}.$$

(d) We use a similar technique as in part (a). With I the integral we seek to compute, then

$$I = \frac{1}{2} \mathrm{Re}\,(J), \qquad\qquad J := \int_{-\infty}^{\infty} e^{i|k|x} x^4 + 1 \, \mathrm{d}x,$$

where we have used the fact that the integrand for I is an even function and is invariant under $k \leftarrow |k|$. Then with C_R as in part (a), the Cauchy Residue Theorem implies,

$$\lim_{R\uparrow\infty} \int_{C_R} f(z) e^{i|k|z} \, \mathrm{d}z + J = 2\pi i \sum_{j=1}^{M} \mathrm{Res}(f(z) e^{i|k|z}; z_j), \tag{4}$$

where

$$f(z) := \frac{1}{z^4 + 1},$$

and $\{z_j\}_{j=1}^{M}$ are the singularities of f in the upper half-plane. We again have that f decays uniformly to 0 as $R \uparrow \infty$:

$$\lim_{R\uparrow\infty} \max_{z \in C_R} |f(z)| \leq \lim_{R\uparrow\infty} \frac{1}{R^4 - 1} = 0,$$

and so by Jordan's Lemma,

$$\lim_{R\uparrow\infty} \int_{C_R} f(z) e^{i|k|z} \, \mathrm{d}z = 0, \quad |k| > 0.$$

The same result is true if $k = 0$ since f is a rational function $f = P/Q$ with $\deg Q \geq \deg P + 2$, i.e., we have

$$\lim_{R\uparrow\infty} \int_{C_R} f(z) e^{i|k|z} \, \mathrm{d}z = 0, \quad |k| \geq 0.$$

There are $M = 2$ singularities of $f(z)e^{i|k|z}$ in the upper half-plane located at $z_1 = e^{i\pi/4}$ and $z_2 = e^{3i\pi/4}$, with residues given by,

$$\begin{aligned} 2\pi i \; \mathrm{Res}(f(z) e^{i|k|z}; z_1) &= 2\pi i \frac{e^{i|k|z_1}}{4z_1^3} = -\frac{i\pi z_1}{2} e^{i|k|z_1}, \\ 2\pi i \; \mathrm{Res}(f(z) e^{i|k|z}; z_2) &= 2\pi i \frac{e^{i|k|z_2}}{4z_2^3} = -\frac{i\pi z_3}{2} e^{i|k|z_3}. \end{aligned}$$

so that (4) becomes,

$$\begin{aligned} J &= -\frac{\pi}{2}\left(iz_1 e^{i|k|z_1} + iz_3 e^{i|k|z_3}\right) \\ &= -\frac{\pi e^{-|k|/\sqrt{2}}}{2}\left(iz_1 e^{i|k|/\sqrt{2}} + iz_3 e^{-i|k|/\sqrt{2}}\right) \\ &= -\frac{\pi e^{-|k|/\sqrt{2}}}{2}\left(e^{i\left(3\pi/4 + |k|/\sqrt{2}\right)} + e^{i\left(5\pi/4 - |k|/\sqrt{2}\right)}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} I = \frac{1}{2}\mathrm{Re}\,(J) &= -\frac{\pi e^{-|k|/\sqrt{2}}}{4}\left(\cos\left(\frac{3\pi}{4} + \frac{|k|}{\sqrt{2}}\right) + \cos\left(\frac{5\pi}{4} - \frac{|k|}{\sqrt{2}}\right)\right) \\ &= \frac{\pi e^{-|k|/\sqrt{2}}}{2\sqrt{2}}\left(\cos\frac{|k|}{\sqrt{2}} + \sin\frac{|k|}{\sqrt{2}}\right) \end{aligned}$$

(g) Since $\sin^4\theta$ has period $\pi/2$, then

$$I = \int_0^{\pi/2} \sin^4\theta \,\mathrm{d}\theta = \frac{1}{4}\int_0^{2\pi} \sin^4\theta\,\mathrm{d}\theta.$$

We now use the parameterization $z = e^{i\theta}$, so that $\sin\theta = \frac{1}{2i}\left(z - 1/z\right)$, yielding,

$$\int_0^{2\pi} \sin^4\theta\,\mathrm{d}\theta = \int_{\partial B_1(0)} \frac{(z^2-1)^4}{16z^5}\frac{\mathrm{d}z}{i}.$$

We compute this latter integral via the Cauchy Residue Theorem:

$$\begin{aligned} \int_{\partial B_1(0)} \frac{(z^2-1)^4}{16z^5}\frac{\mathrm{d}z}{i} &= \frac{2\pi i}{16 i}\,\mathrm{Res}\left(\frac{(z^2-1)^4}{z^5}; 0\right) \\ &= \frac{\pi}{8}\frac{1}{4!}\left(\frac{\mathrm{d}^4}{\mathrm{d}z^4}(z^2-1)^4\right)\Big|_{z=0} \\ &= \frac{\pi}{8(4!)}\frac{\mathrm{d}^4}{\mathrm{d}z^4}\left(z^8 - 4z^6 + 6z^4 - 4z^2 + 1\right)\Big|_{z=0} = \frac{3\pi}{4}. \end{aligned}$$

Thus,

$$I = \frac{1}{4}\frac{3\pi}{4} = \frac{3\pi}{16}$$

4.2.5. Consider a rectangular contour with corners at $b \pm iR$ and $b + 1 \pm iR$. Use this contour to show that,

$$\lim_{R\to\infty} \frac{1}{2\pi i}\int_{b-iR}^{b+iR} \frac{e^{az}}{\sin \pi z}\,\mathrm{d}z = \frac{1}{\pi(1+e^{-a})},$$

where $0 < b < 1$ and $|\mathrm{Im}\,(a)\,| < \pi$.

Solution: For finite R, let the left, right, bottom, and top sides of the rectange be denote C_ℓ, C_r, C_b, and C_t, respectively. The integrand has singularities at $z = n$, $n \in \mathbb{Z}$, which are all simple poles, but $z = 1$ is the only singularity lying inside this contour. Therefore,

$$\operatorname{Res}\left(\frac{e^{az}}{\sin \pi z}; 1\right) = \frac{e^a}{\pi \cos \pi} = -\frac{e^a}{\pi}.$$

Letting,

$$f(z) = \frac{e^{az}}{\sin \pi z},$$

then

$$\begin{aligned}
\left|\int_{C_b} f(z)\,\mathrm{d}z\right| &= \left|\int_b^{b+1} \frac{e^{a(x-iR)}}{\sin \pi(x-iR)}\,\mathrm{d}x\right| \\
&\leq \int_b^{b+1} \left| 2i \frac{e^{x\mathrm{Re}(a)+R\mathrm{Im}(a)} e^{i(x\mathrm{Im}(a)-R\mathrm{Re}(a))}}{e^{i\pi x+\pi R} - e^{-\pi R - i\pi x}}\right| \mathrm{d}x \\
&= 2e^{R\mathrm{Im}(a)} \int_b^{b+1} \frac{e^{x\mathrm{Re}(a)}}{\left|e^{i\pi x+\pi R} - e^{-\pi R-i\pi x}\right|}\,\mathrm{d}x \\
&\leq \frac{2e^{R\mathrm{Im}(a)}}{e^{\pi R}} \frac{1}{1-e^{-2\pi R}} \int_b^{b+1} e^{x\mathrm{Re}(a)}\,\mathrm{d}x \\
&\leq 2e^{R(\mathrm{Im}(a)-\pi)} \frac{\max\{e^{b\mathrm{Re}(a)}, e^{(b+1)\mathrm{Re}(a)}\}}{1-e^{-2\pi R}}.
\end{aligned}$$

Therefore, taking the limit in R and noting that $\mathrm{Im}\,(a) - \pi < 0$, then,

$$\lim_{R\uparrow\infty} \int_{C_b} f(z)\,\mathrm{d}z = 0.$$

A similar computation can be carried out for C_t by simply performing the same computation as on C_b but by making the replacement $R \leftarrow -R$:

$$\begin{aligned}
\left|\int_{C_t} f(z)\,\mathrm{d}z\right| &\leq 2e^{-R\mathrm{Im}(a)} \int_b^{b+1} \frac{e^{x\mathrm{Re}(a)}}{\left|e^{i\pi x-\pi R} - e^{\pi R-i\pi x}\right|}\,\mathrm{d}x \\
&\leq 2\frac{e^{-R\mathrm{Im}(a)}}{e^{\pi R}} \frac{1}{1-e^{-2\pi R}} \int_b^{b+1} e^{x\mathrm{Re}(a)}\,\mathrm{d}x \\
&\leq 2e^{R(-\mathrm{Im}(a)-\pi)} \frac{\max\{e^{b\mathrm{Re}(a)}, e^{(b+1)\mathrm{Re}(a)}\}}{1-e^{-2\pi R}}
\end{aligned}$$

We also have $-\mathrm{Im}\,(a) - \pi < 0$, so taking limits in R yields:

$$\lim_{R\uparrow\infty} \int_{C_t} f(z)\,\mathrm{d}z = 0.$$

On the left contour, C_ℓ, we have,

$$\begin{aligned}\int_{C_\ell} f(z) &= \int_{b+iR}^{b-iR} f(z)\,\mathrm{d}z \\ &= \int_{R}^{-R} f(b+iy) i\,\mathrm{d}y \\ &= -\int_{-R}^{R} f(b+iy) i\,\mathrm{d}y \\ &= -\int_{b-iR}^{b+iR} f(z)\,\mathrm{d}z =: -2\pi i\, I(R),\end{aligned}$$

i.e., we have defined

$$I(R) \coloneqq \frac{1}{2\pi i}\int_{b-iR}^{b+iR} \frac{e^{az}}{\sin \pi z}\,\mathrm{d}z.$$

On the right contour C_r we have,

$$\begin{aligned}\int_{C_r} f(z) &= \int_{b+1-iR}^{b+1+iR} \frac{e^{az}}{\sin \pi z}\,\mathrm{d}z \\ &\overset{w=z-1}{=} e^a \int_{b-iR}^{b+iR} \frac{e^{aw}}{\sin(\pi w - \pi)}\,\mathrm{d}w \\ &= -e^a \int_{b-iR}^{b+iR} \frac{e^{aw}}{\sin \pi w}\,\mathrm{d}w = -e^a(2\pi i) I(R).\end{aligned}$$

Then the Cauchy Residue Theorem states,

$$\int_{C_b} f(z)\,\mathrm{d}z + \int_{C_t} f(z)\,\mathrm{d}z + \int_{C_\ell} f(z)\,\mathrm{d}z + \int_{C_r} f(z)\,\mathrm{d}z = 2\pi i \mathrm{Res}(f;1),$$

and using all our computations above yields,

$$\int_{C_b} f(z)\,\mathrm{d}z + \int_{C_t} f(z)\,\mathrm{d}z + 2\pi i\left(-1 - e^a\right) I(R) = -2\pi i \frac{e^a}{\pi}$$

Taking limits in R:

$$\lim_{R\uparrow\infty} I(R) = \frac{e^a}{\pi(1+e^a)} = \frac{1}{\pi(1+e^{-a})},$$

which is what we wished to show.

4.2.7. Use a sector contour with radius R, as in Figure 4.2.6 in the text, centered at the origin with angle $0 \le \theta \le \frac{2\pi}{5}$ to find, for $a > 0$,

$$\int_0^\infty \frac{\mathrm{d}x}{x^5 + a^5} = \frac{\pi}{5a^4 \sin\frac{\pi}{5}}.$$

Solution: We use the Cauchy Residue Theorem, and so proceed to define and integrate along a closed contour. The contour we consider contains two rays of length R, one extending from the origin at angle 0, and the second extending from the origin at angle $\frac{2\pi}{5}$. We denote these two contours by C_0 (angle 0) and C_+ (angle $2\pi/5$), respectively. We will call the circular arc of radius R connecting these as C_R. Defining,

$$f(z) = \frac{1}{z^5 + a^5},$$

which satisfies,

$$\lim_{R\to\infty} \max_{z \in C_R} |z f(z)| = \lim_{R\to\infty} \max_{z \in C_R} \frac{R}{|z^5 + a^5|} \le \lim_{R\to\infty} \max_{z \in C_R} \frac{R}{R^5 - a^5} = 0,$$

then we have,

$$\lim_{R\to\infty} \int_{C_R} f(z)\,\mathrm{d}z = 0.$$

Along the contour C_0, through the parameterization $z = x$ as x ranges from 0 to R, we have,

$$\lim_{R\to\infty} \int_{C_0} f(z)\,\mathrm{d}z = \int_0^\infty \frac{\mathrm{d}x}{x^5 + a^5} =: I.$$

Along the contour C_+, through the parameterization $z = re^{2\pi i/5}$, as r ranges from R to 0, we have,

$$\lim_{R\to\infty} \int_{C_+} f(z)\,\mathrm{d}z = \int_\infty^0 \frac{e^{2\pi i/5}\,\mathrm{d}r}{r^5 + a^5} = -e^{2\pi i/5} I$$

Finally, the singularities of f are all simple poles at the points,

$$z = z_j := a^{1/5} e^{i\pi/5} e^{i2\pi j/5}, \qquad j = 0, 1, 2, 3, 4,$$

and only one of these poles, z_0, lies inside the contour. Its corresponding residue is,

$$2\pi i \,\mathrm{Res}(f; z_0) = \frac{2\pi i}{5 z_0^4} = \frac{2\pi i}{5a^4} e^{-4\pi i/5}$$

Finally, the Cauchy Residue Theorem integrating over C_0, C_R, and C_+, after taking the limit $R \to \infty$, reads,

$$I + 0 - e^{2\pi i/5} I = \frac{\pi}{5a^4} 2ie^{-4\pi i/5}$$

Rearranging, this yields,

$$\begin{aligned} I &= \frac{\pi}{5a^4} \frac{2ie^{-4\pi i/5}}{e^{i\pi/5}\left(e^{-i\pi/5} - e^{i\pi/5}\right)} \\ &= \frac{\pi}{5a^4} \frac{-2i}{e^{-i\pi/5} - e^{i\pi/5}} \\ &= \frac{\pi}{5a^4} \frac{1}{\sin \frac{\pi}{5}}, \end{aligned}$$

which is what we wanted to show.