Department of Mathematics, University of Utah

Applied Complex Variables and Asymptotic Methods

MATH 6720 – Section 001 – Spring 2023

Homework 3

Taylor Series

Due: Tuesday, February 28, 2023

Below, problem C in section A.B is referred to as exercise A.B.C.
Text: *Complex Variables: Introduction and Applications*, Ablowitz & Fokas,

Exercises: 2.6.2
2.6.5
2.6.7
3.2.3
3.3.3
3.3.4
3.5.1, parts a) - f), and i) and j)
3.5.2, parts a) - c)

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2.6.2. Evaluate the integrals $\oint_C f(z)\,\mathrm{d}z$ over a contour C, where C is the boundary of a square with diagonal opposite corners at $z = -(1+i)R$ and $z = (1+i)R$, where $R > a > 0$, and where $f(z)$ is given by the following (use Eq. (1.2.19) in the text as necessary):

(a) $\frac{e^z}{z - \frac{\pi i}{4}a}$
(b) $\frac{e^z}{\left(z - \frac{\pi i}{4}a\right)^2}$
(c) $\frac{z^2}{2z+a}$
(d) $\frac{\sin z}{z^2}$
(e) $\frac{\cosh z}{z}$

Solution: Our main tool in this exercise is the Cauchy integral formula (CIF).

(a) Let $f(z) = e^z$, which is entire. The point $\frac{\pi i}{4}a$ lies inside the square C, so the CIF states,

$$\oint_C \frac{e^z}{z - \frac{\pi i}{4}a}\,\mathrm{d}z = \oint_C \frac{f(z)}{z - \frac{\pi i}{4}a}\,\mathrm{d}z = 2\pi i f\left(\frac{\pi i}{4}a\right) = 2\pi i e^{ai\pi/4}.$$

(b) With the same $f(z) = e^z$ as above, the CIF for derivatives of f implies,

$$\oint_C \frac{e^z}{\left(z - \frac{\pi i}{4}a\right)^2}\,\mathrm{d}z = \oint_C \frac{f(z)}{\left(z - \frac{\pi i}{4}a\right)^2}\,\mathrm{d}z = 2\pi i f'\left(\frac{\pi i}{4}a\right) = 2\pi i e^{ai\pi/4}.$$

(c) We define $f(z) = \frac{z^2}{2}$, which is entire, so that,

$$\oint_C \frac{z^2}{2z+a}\,\mathrm{d}z = \oint_C \frac{f(z)}{z + a/2}\,\mathrm{d}z \stackrel{\text{CIF}}{=} 2\pi i f(-a/2) = \frac{a^2\pi i}{4},$$

where we have also used the fact that $-a/2$ lies inside C.

(d) Defining $f(z) = \sin z$, which is entire, then

$$\oint_C \frac{\sin z}{z^2}\,\mathrm{d}z = \oint_C \frac{f(z)}{z^2}\,\mathrm{d}z \overset{\text{CIF}}{=} 2\pi i f'(0) = 2\pi i$$

(e) Defining $f(z) = \cosh z$, which is entire, we have,

$$\oint_C \frac{\cosh z}{z}\,\mathrm{d}z = \oint_C \frac{f(z)}{z}\,\mathrm{d}z \overset{\text{CIF}}{=} 2\pi i f(0) = 2\pi i$$

2.6.5. Consider two entire functions with no zeros and having a ratio equal to unity at infinity. Use Liouville's Theorem to show that they are in fact the same function.

Solution: Let f_1 and f_2 be the functions in question, and define,

$$g(z) = \frac{f_1(z)}{f_2(z)},$$

which itself is entire since f_2 has no zeros. Since $\lim_{z\to\infty} g(z) = 1$, then there is some $R \geq 0$ such that

$$|z| > R \implies |g(z) - 1| < \frac{1}{2},$$

which in particular means that,

$$|z| > R \implies |g(z)| < \frac{3}{2}. \tag{1a}$$

Now on $\overline{B_R(0)}$ (the closed origin-centered ball of radius R), g is analytic, and in particular continuous over this closed and bounded set, so that

$$M := \max_{z\in\overline{B_R(0)}} |g(z)| < \infty, \tag{1b}$$

i.e., g is bounded on $\overline{B_R(0)}$. Combining (1a) and (1b) implies,

$$\max_{z\in\mathbb{C}} |g(z)| \leq \max\left\{\frac{3}{2}, M\right\} < \infty,$$

i.e., g is bounded on $\mathbb{C}$ and is analytic on $\mathbb{C}$. By Liouville's theorem, $g(z)$ is constant, and in particular $\lim_{z\to\infty} g(z) = 1$ implies that $g(z) = 1$ over $\mathbb{C}$, i.e., $f_1(z) = f_2(z)$ over $\mathbb{C}$.

2.6.7. Let $f(z)$ be an entire function, with $|f(z)| \leq C|z|$ for all z, where C is a constant. Show that $f(z) = Az$, where A is a constant.

Solution: Our main goal will be to show that $f'' \equiv 0$. Fix an arbitrary $z \in \mathbb{C}$, and let C be a z-centered circle of radius $R > |z|$. The Cauchy Integral formula for the second derivative of f reads,

$$f''(z) = \frac{2!}{2\pi i}\oint_C \frac{f(w)}{(w-z)^3}\,\mathrm{d}w.$$

Using $|f(w)| \leq C|w|$ and parameterizing the integral over the circle with $w(\theta) = z + Re^{i\theta}$, we have,

$$\begin{aligned}
\left|f''(z)\right| &\leq \frac{1}{\pi i} \oint_C \frac{|f(w)|}{|w-z|^3} |\,\mathrm{d}w| \\
&\leq \frac{C}{\pi i} \oint_C \frac{|w|}{|w-z|^3} |\,\mathrm{d}w| \\
&= \frac{C}{\pi i} \int_0^{2\pi} \frac{|z + Re^{i\theta}|}{R^3} R \,\mathrm{d}\theta \\
&\overset{|z|<R}{\leq} \frac{C}{\pi R^2} \int_0^{2\pi} 2R \,\mathrm{d}\theta = \frac{4C}{R}
\end{aligned}$$

This bound is true for every $R > |z|$, i.e., $|f''(z)| < \epsilon$ for every $\epsilon > 0$, implying that $|f''(z)| = f''(z) = 0$. Since z was arbitrary, we have $f''(z) = 0$ for all $z \in \mathbb{C}$. Therefore, $f(z) = Az + B$ for some constants A and B.
The constant B must be zero since $|f(z)| \leq C|z|$ implies that $|f(0)| \leq 0$, i.e., $|B| \leq 0$, so $B = 0$. Hence, $f(z) = Az$.

3.2.3. Let the Euler number E_n be defined by the power series,

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n.$$

(a) Find the radius of convergence of this series.
(b) Determine the first six Euler numbers.

Solution:

(a) Since 1 and $\cosh z$ are both entire functions, then $1/\cosh z$ fails to be analytic only where $\cosh z = 0$. The roots of this function correspond to the roots of the equation,

$$e^z + e^{-z} = 0 \quad \xrightarrow{e^z \neq 0} \quad e^{2z} + 1 = 0.$$

i.e., $e^{2z} = -1$. Writing this in terms of logarithms, we have,

$$z = \frac{1}{2} \log -1 = \frac{1}{2}\left(-i\pi + i2\pi k\right) = -\frac{i\pi}{2} + ik\pi,$$

for every $k \in \mathbb{Z}$. In particular, the two roots that are closest to the origin are,

$$z = \pm \frac{i\pi}{2}.$$

In other words, $1/\cosh z$ is analytic on $|z| < R_0$ for every $R_0 < \pi/2$. Hence, the radius of convergence for this power/Taylor series is $R = \pi/2$.

(b) Within the region of convergence of the series, we rewrite it as,

$$\begin{aligned}
1 &= \cosh z \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n \\
&= \left(\sum_{k=0}^{\infty} c_k z^k\right)\left(\sum_{n=0}^{\infty} \frac{E_n}{n!} z^n\right) \\
&= \sum_{n=0}^{\infty} d_n z^n,
\end{aligned}$$

where

$$c_k = \begin{cases} \frac{1}{k!}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases} \qquad\qquad d_n = \sum_{k=0}^{n} \frac{E_k}{k!} c_{n-k}.$$

Note that since this must be the power series for the function 1, then

$$d_n = \begin{cases} 1, & n = 0 \\ 0, & n \geq 1. \end{cases}$$

Hence, we can determine the first 6 Euler numbers by equating the two previous expressions for d_n:

$$\begin{aligned}
1 = d_0 &= \frac{E_0}{0!} c_0 = E_0, \\
0 = d_1 &= \frac{E_0}{0!} c_1 + \frac{E_1}{1!} c_0 = E_1 \\
0 = d_2 &= \frac{E_0}{0!} c_2 + \frac{E_1}{1!} c_1 + \frac{E_2}{2!} c_0 = \frac{E_0}{2} + \frac{E_2}{2} \\
0 = d_3 &= \frac{E_0}{0!} c_3 + \frac{E_1}{1!} c_2 + \frac{E_2}{2!} c_1 + \frac{E_3}{3!} c_0 = \frac{E_1}{2} + \frac{E_3}{6} \\
0 = d_4 &= \frac{E_0}{0!} c_4 + \frac{E_1}{1!} c_3 + \frac{E_2}{2!} c_2 + \frac{E_3}{3!} c_1 + \frac{E_4}{4!} c_0 = \frac{E_0}{24} + \frac{E_2}{4} + \frac{E_4}{24} \\
0 = d_5 &= \frac{E_0}{0!} c_5 + \frac{E_1}{1!} c_4 + \frac{E_2}{2!} c_3 + \frac{E_3}{3!} c_2 + \frac{E_4}{4!} c_1 + \frac{E_5}{5!} c_0 = \frac{E_1}{24} + \frac{E_3}{12} + \frac{E_5}{120}
\end{aligned}$$

This is a lower triangular linear system for the unknowns $(E_0, E_1, E_2, E_3, E_4, E_5)$, whose solution is:

$$\begin{pmatrix} E_0 \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 5 \\ 0 \end{pmatrix}$$

3.3.3. Given the function

$$f(z) = \frac{z}{(z-2)(z+i)},$$

expand $f(z)$ in a Laurent series in powers of z in the regions,

(a) $|z| < 1$
(b) $1 < |z| < 2$
(c) $|z| > 2$

Solution: Before delving into the individual parts of this problem, we perform some preliminary computations. First, expanding f in partial fractions yields the representation,

$$f(z) = \frac{\frac{2}{2+i}}{z-2} + \frac{\frac{i}{2+i}}{z+i}, \tag{2}$$

so that we can accomplish this problem by considering Laurent series' for $1/(z-2)$ and $1/(z+i)$. For the first expansion, we note:

$$|z| < 2 \;:\; \frac{1}{z-2} = \left(-\frac{1}{2}\right)\frac{1}{1-(z/2)} \overset{|z/2|<1}{=} -\frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{z}{2}\right)^j = \sum_{j=0}^{\infty} -\frac{1}{2^{j+1}}z^j,$$

$$|z| > 2 \;:\; \frac{1}{z-2} = \left(\frac{1}{z}\right)\frac{1}{1-(2/z)} \overset{|2/z|<1}{=} \frac{1}{z}\sum_{j=0}^{\infty}\left(\frac{2}{z}\right)^j = \sum_{j=-\infty}^{-1} \frac{1}{2^{j+1}}z^j,$$

and this implies that,

$$\frac{1}{z-2} = \begin{cases} \displaystyle\sum_{j=0}^{\infty} a_j z^j, & |z| < 2, \\ \displaystyle\sum_{j=-\infty}^{-1} a_j z^j, & |z| > 2, \end{cases} \qquad a_j = \begin{cases} -\frac{1}{2^{j+1}}, & j \geq 0 \\ \frac{1}{2^{j+1}}, & j < 0. \end{cases} \tag{3}$$

Similarly, for $1/(z+i)$ we have,

$$|z| < 1 \;:\; \frac{1}{z+i} = \left(\frac{1}{i}\right)\frac{1}{1-iz} \overset{|iz|<1}{=} \frac{1}{i}\sum_{j=0}^{\infty}(iz)^j = \sum_{j=0}^{\infty} i^{j-1}z^j,$$

$$|z| > 1 \;:\; \frac{1}{z+i} = \left(\frac{1}{z}\right)\frac{1}{1-(1/iz)} \overset{|iz|>1}{=} \frac{1}{z}\sum_{j=0}^{\infty}\frac{1}{(iz)^j} = \sum_{j=-\infty}^{-1} i^{j+1}z^j,$$

and so,

$$\frac{1}{z+i} = \begin{cases} \displaystyle\sum_{j=0}^{\infty} b_j z^j, & |z| < 1, \\ \displaystyle\sum_{j=-\infty}^{-1} b_j z^j, & |z| > 1, \end{cases} \qquad b_j = \begin{cases} i^{j-1}, & j \geq 0 \\ i^{j+1}, & j < 0. \end{cases} \tag{4}$$

We can now answer the individual parts of this question:

(a) For $|z| < 1$, we use the Taylor series both from (3) and (4) in (2) to obtain,

$$f(z) = \sum_{j=0}^{\infty} c_j z^j, \qquad c_j = \frac{2a_j}{2+i} + \frac{ib_j}{2+i} = \frac{1}{2+i}\left(-\frac{1}{2^j} + i^j\right), \quad j \geq 0.$$

(b) For $1 < |z| < 2$, we use the Laurent series from (3) and the Taylor series from (4), obtaining,

$$f(z) = \sum_{j=-\infty}^{\infty} c_j z^j, \qquad c_j = \begin{cases} \frac{2a_j}{2+i} = -\frac{1}{(2+i)2^j}, & j \geq 0, \\ \frac{ib_j}{2+i} = \frac{i^{j+2}}{2+i}, & j < 0 \end{cases}$$

(c) For $|z| > 2$, we exercise the Laurent series from both (3) and (4) to obtain,

$$f(z) = \sum_{j=-\infty}^{-1} c_j z^j, \qquad c_j = \frac{2a_j}{2+i} + \frac{ib_j}{2+i} = \frac{1}{2+i}\left(\frac{1}{2^j} - i^j\right), \quad j < 0$$

3.3.4. Evaluate the integral $\oint_C f(z)\,\mathrm{d}z$ where C is the unit circle centered at the origin and $f(z)$ is given as follows,
(a) $\frac{e^z}{z^3}$
(b) $\frac{1}{z^2 \sin z}$
(c) $\tanh z$
(d) $\frac{1}{\cos 2z}$
(e) $e^{1/z}$

Solution: Assuming f is analytic in some open domain of $|z| = 1$, our strategy in this problem is to utilize the Laurent series formula,

$$g(z) = \sum_{n \in \mathbb{Z}} c_n z^n, \qquad c_n = \frac{1}{2\pi i} \oint_C \frac{g(z)}{z^{n+1}}\,\mathrm{d}z,$$

so that, by appropriately defining g, integrals around C can be computed by identifying certain coefficients of the Laurent series of g. (Note that this is equivalent to expanding f in a Laurent series and computing the c_{-1} coefficient for f.)

(a) With $g(z) = e^z$, then

$$\oint_C f(z)\,\mathrm{d}z = \oint_C \frac{g(z)}{z^3}\,\mathrm{d}z = 2\pi i c_2.$$

The function g has the convergent Taylor series $g(z) = 1+z+z^2/2+\cdots$, so we immediately conclude that $c_2 = 1/2$, and thus, $\oint_C f(z)\,\mathrm{d}z = \pi i$.

(b) We take $g(z) = z/(\sin z)$, so that,

$$\oint_C f(z)\,\mathrm{d}z = \oint_C \frac{g(z)}{z^3}\,\mathrm{d}z = 2\pi i c_2.$$

The Laurent series expansion around $z = 0$ for g, which is analytic around $|z| = 1$, can be computed via,

$$\frac{z}{\sin z} = \frac{z}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots} = \frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots} = \frac{1}{1 - \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots\right)}$$
$$= 1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots\right) + \cdots,$$

where the last equality uses the geometric series on the terms in parenthesis. From this expression, we see that $c_2 = 1/3! = 1/6$, so that $\oint_C f(z)\,\mathrm{d}z = \pi i/3$.

(c) The function $\tanh z$ is analytic everywhere except where the denominator ($\cosh z$) vanishes. The denominator vanishes at $z = (2j+1)i\pi/2$, for $j \in \mathbb{Z}$. Note that C does not enclose any of these points, and so f is analytic inside C. By the Cauchy-Goursat theorem, $\oint_C f(z)\,\mathrm{d}z = 0$.

(d) For this problem, we cannot directly compute a Laurent series since $\cos 2z = 0$ for $z = \pm\frac{\pi}{4}$, which both lie within C. Instead, we deform C into two circles of small radius (say radius smaller than $\frac{\pi}{4}$) with a crosscut connecting them, with the first circle C_- counterclockwise around $z = -\pi/4$, and second circle C_+ counterclockwise around $\pi/4$. The two crosscut integrals add-out, and so we must compute,

$$\int_C \frac{1}{\cos 2z}\,\mathrm{d}z = \int_{C_-} \frac{1}{\cos 2z}\,\mathrm{d}z + \int_{C_+} \frac{1}{\cos 2z}\,\mathrm{d}z.$$

For the C_- integral, $g_-(z) := \frac{1}{\cos 2z}$ is analytic in a neighborhood of $z = -\pi/4$, and we have,

$$g_-(z) = \frac{1}{\cos 2z} = \frac{1}{\sin(2(z+\pi/4))} = \frac{1}{2(z+\pi/4)} \sum_{j\geq 0} \left[\frac{(2(z+\pi/4))^2}{3!} - \frac{(2(z+\pi/4))^4}{5!} - \ldots \right]^j$$

where the last equality is the Laurent series of $1/\sin w$ around $w = 0$ with $w \leftarrow 2(z+\pi/4)$. (The computation uses the Taylor expansion of $\sin w$ along with a geometric series.) Hence, the Laurent series coefficient c_{-1} for g_- around $z = -\pi/4$ is given by $c_{-1} = \frac{1}{2}$, so that,

$$\int_{C_-} \frac{1}{\cos 2z} \, \mathrm{d}z = \frac{1}{2}.$$

A similar computation with,

$$g_+(z) := \frac{1}{\cos 2z} = -\frac{1}{\sin(2(z-\pi/4))} = \frac{-1}{2(z-\pi/4)} \sum_{j\geq 0} \left[\frac{(2(z-\pi/4))^2}{3!} - \frac{(2(z-\pi/4))^4}{5!} - \ldots \right]^j,$$

implies that

$$\int_{C_+} \frac{1}{\cos 2z} \, \mathrm{d}z = -\frac{1}{2}.$$

Hence,

$$\int_C \frac{1}{\cos 2z} \, \mathrm{d}z = \int_{C_-} \frac{1}{\cos 2z} \, \mathrm{d}z + \int_{C_+} \frac{1}{\cos 2z} \, \mathrm{d}z = \frac{1}{2} - \frac{1}{2} = 0$$

(e) Again defining $g = f$, we have

$$\oint_C f(z) \, \mathrm{d}z = 2\pi i c_{-1}.$$

The Laurent series for $f = e^{1/z}$ is given by,

$$e^{1/z} \overset{z=1/w}{=} e^w = \sum_{j\geq 0} \frac{w^j}{j!} \overset{z=1/w}{=} \sum_{j\leq 0} \frac{z^j}{(-j)!} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots,$$

so that $c_{-1} = 1$, and hence $\oint_C f(z) \, \mathrm{d}z = 2\pi i$.

3.5.1. Discuss the type of singularity (removable, pole and order, essential, branch, cluster, natural barrier, etc.); if the type is a pole give the strength of the pole, and give the nature (isolated or not) of all singular points associated with the following functions. Include the point at infinity.

(a) $\frac{e^{z^2}-1}{z^2}$

(b) $\frac{e^{2z}-1}{z^2}$

(c) $e^{\tan z}$

(d) $\frac{z^3}{z^2+z+1}$

(e) $\frac{z^{1/3}-1}{z-1}$

(f) $\log(1 + z^{1/2})$

(i) $\operatorname{sech} z$

(j) $\coth 1/z$

Solution:

(a) The point $z = 0$ is a *removable (hence isolated) singularity*, and $z = \infty$ is an *essential (hence isolated) singularity.* The finite-z singularities only occur when the denominator vanishes, i.e., $z = 0$, where f (and also f') are undefined. Through a Taylor expansion of the numerator, we find,

$$\frac{e^{z^2}-1}{z^2} = \frac{1}{z^2}\left[-1 + \sum_{j=0}^{\infty} \frac{z^{2j}}{j!}\right] = \sum_{j=0}^{\infty} \frac{z^{2j}}{(j+1)!},$$

hence this function has a convergent power series around $z = 0$, and thus $z = 0$ is a removable singularity. Redefining the function to have value 1 at $z = 0$ makes it analytic. To investigate $z = \infty$, consider the expansion above in the variable $w = 1/z$. Then as a function of w this function has an infinite principal part of its Laurent expansion around $w = 0$, so that $w = 0$ (hence $z = \infty$) is an essential singularity.

(b) The point $z = 0$ is a *simple pole (hence isolated)* with strength 2, and $z = \infty$ is an *essential (isolated) singularity.* Again, we take a Taylor series expansion and simplify:

$$\frac{e^{2z}-1}{z^2} = \frac{1}{z^2}\left[-1 + \sum_{j=0}^{\infty} \frac{(2z)^j}{j!}\right] = \frac{2}{z} + 2 + \ldots,$$

where $\ldots$ is a power series around $z = 0$. Thus, $z = 0$ is a simple pole of strength 2. Making the definition $w = 1/z$, we see that the again the the principal part of the expansion above in terms of w has an infinite number of terms, so that $w = 0$ (hence $z = \infty$) is an essential singularity.

(c) The point $z = \infty$ is a *cluster (non-isolated) singular point*, and $z = \frac{(2n+1)\pi}{2}$ for $n \in \mathbb{Z}$ are *essential (isolated) singularities*. Making the substituion $w = 1/z$, then $\exp(\tan 1/w)$ has singularities for $w = \frac{2}{(2n+1)}$, $n \in \mathbb{Z}$, which cluster around 0, so $w = 0$ ($z = \infty$) is a cluster singularity. To establish that the finite z points are essential singularities, consider $z = -\pi/2$, and note that in a neighborhood around this point,

$$\exp(\tan z) = \exp\left(\sin z \frac{1}{\sin(z+\pi/2)}\right) = \exp\left(\frac{\sin z}{z+\pi/2} \sum_{j\geq 0}\left[\frac{(z+\pi/2)^2}{3!} - \frac{(z+\pi/2)^4}{5!} + \ldots\right]^j\right),$$

and hence around $z = -\pi/2$, this function behaves like $\exp(1/z)$ around $z = 0$, i.e., as an essential singularity.

(d) The point $z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ is a *simple pole (isolated)* with strength $-i/\sqrt{3}$, the point $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ is a *simple pole (isolated)* with strength $i/\sqrt{3}$, and $z = \infty$ is a *simple pole (isolated)* of strength 1. The point at $z = \infty$ is the simplest to see with $w = 1/z$ and since for large z, $\frac{z^3}{z^2+z+1} \sim z = \frac{1}{w}$, which is a simple pole of strength 1. The other two poles are found at the roots of the denominator:

$$z^2 + z + 1 = 0 \implies z = z_0, \overline{z}_0 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Hence, the function can be rewritten as,

$$\frac{z^3}{z^2+z+1} = \frac{z^3}{(z-z_0)(z-\overline{z}_0)}.$$

corresponding to simple poles at $z = z_0, \overline{z}_0$. The strength of the pole at $z = z_0$ is given by,

$$\frac{z_0^3}{z_0 - \overline{z}_0} = -\frac{i}{\sqrt{3}} z_0^3 = -\frac{i}{\sqrt{3}} z_0^2 z_0 \overset{z_0^2+z_0+1=0}{=} -\frac{i}{\sqrt{3}}(-1-z_0)z_0 = -\frac{i}{\sqrt{3}}(-z_0 - z_0^2) = -\frac{i}{\sqrt{3}}$$

A similar computation can be carried out for the pole at $\overline{z}_0$, showing its strength is $+i/\sqrt{3}$.

(e) The points $z = 0, \infty$ are each a *branch (non-isolated) singularity.* For finite z, this function has three branches. For two of the branches, $z = 1$ is a *simple pole (isolated) with strengths* $-\frac{3}{2} + i\sqrt{3}/2$ *and* $-\frac{3}{2} - i\sqrt{3}/2$, respectively. The third branch has a *removable (isolated) singularity at* $z = 1$. To see the branch point at ∞, make the substitution $w = 1/z$, so that,

$$\frac{z^{1/3} - 1}{z - 1} = \frac{w(w^{-1/3} - 1)}{1 - w}.$$

Thus, we see through the $w^{-1/3}$ term that $w = 0$ ($z = \infty$) is a branch point singularity for this function. The point $z = 0$ is a branch point as can be seen directly by definition of the function. To assess the situation around $z = 1$, we make the branch cut of the function along the negative real axis, corresponding to three branches:

- Branch 1: $z^{1/3} = |z|e^{i(\arg z)/3}$ for $\arg z \in [-\pi, \pi)$
- Branch 2: $z^{1/3} = |z|e^{i(\arg z)/3}$ for $\arg z \in [\pi, 3\pi)$
- Branch 3: $z^{1/3} = |z|e^{i(\arg z)/3}$ for $\arg z \in [3\pi, 5\pi)$

On branch 1 around $z = 1$, we have $z^{1/3} \to z^{1/3}$ for $\arg z$ around 0. Using the Taylor expansion of $z^{1/3}$ around $z = 1$, we have,

$$\frac{-1 + z^{1/3}}{z - 1} = \frac{-1 + 1 + \frac{1}{3}(z-1) + \ldots}{z - 1} = \frac{1}{3} + \ldots,$$

where $\ldots$ denote higher-order power series terms. Hence, $z = 1$ for Branch 1 a removable singularity, and defining the function to have value $1/3$ at $z = 1$ makes it analytic around $z = 1$. On branch 2, we have $z^{1/3} \to e^{i2\pi/3} z^{1/3} = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z^{1/3}$ for $\arg z$ around 0. In this case, the numerator is analytic around $z = 1$ but does not vanish there (instead taking on value $-3/2 + i\sqrt{\frac{3}{2}}$), so we conclude that $z = 1$ is a simple pole with strength $-3/2 + i\sqrt{\frac{3}{2}}$. A similar argument can be made for branch 3.

(f) The points $z = 0, \infty$ are *branch (non-isolated) points*, and $z = 1$ is a *branch (non-isolated)* point as well. That $z = 0, \infty$ are branch points is a direct result from the fact that these are branch points for $z \mapsto \sqrt{z}$. To see that $z = 1$ is a branch point, consider the (non-principal) branch of $\sqrt{z}$, given by $-\sqrt{z}$ for $\arg z \in [-\pi, \pi)$. Then,

$$\log(1 + \sqrt{z}) \to \log(1 - \sqrt{z}),$$

so that $z = 1$ is also a branch point since $\log w$ has a branch point at $w = 0$. No additional branch points are given by the principal branch of $\sqrt{z}$ since $1 + \sqrt{z}$ cannot vanish on the principal branch of $\sqrt{z}$.

(i) $z = \infty$ is a *cluster point (non-isolated) singularity*, and $z = i(2n + 1)\pi/2$ are *simple poles either of strength* $+1$ *or* -1. To see that $z = \infty$ is a cluster point, note that $\cosh(1/w)$ has infinitely many zeros near $w = 0$. To establish that the other, finite points are simple poles, note that $\cosh z = \cos iz$ has roots at $z = i(2n + 1)\pi/2$, and $(\cos iz)/(z - i(2n + 1)\pi/2)$ has a limit at $z = i(2n + 1)\pi/2$ either $\pm i$, so that $\operatorname{sech} z$ has a simple pole there of strength $\pm i$. In particular, the pole at $z = i(2n + 1)\pi/2$ has strength $i(-1)^{n+1}$.

(j) $z = 0$ is a *cluster (non-isolated) point*, $z = \infty$ is a *simple pole* with strength 1, and $z = \frac{i}{n\pi}$ for $n \in \mathbb{Z}\backslash\{0\}$ are *simple poles of strength* ± 1. To see that $z = 0$ is a cluster point, note that $\sinh w$ vanishes infinitely often for w near infinity, so $\coth 1/z$ has infinitely many singularities near $z = 0$. That $z = \infty$ is a simple pole can be understood since with $w = 1/z$, then $\coth 1/z = \coth w = (\cosh w)/(\sinh w)$, and $\sinh w$ vanishes at 0. To identify that this is a pole, note that around $w = 0$,

$$\sinh w = -i \sin(iw) = -i \left(iw - \frac{(iw)^3}{3!} + \dots \right) = w + \dots,$$

where $\dots$ denote higher order power series terms. Hence, $\coth w$ behaves like $\cosh(0)/w$ near $w = 0$, hence the strength of the $w = 0$ pole is 1. Similarly, $z = i/(n\pi)$ for $n \in \mathbb{Z}\backslash\{0\}$ are where $\sinh 1/z$ vanish linearly, and hence are where $\coth z$ has simple poles of strength $1/(n\pi)^2$, as can be seen by computing the limit of $(z - i/(n\pi))\coth(1/z)$ as $z \to i/(n\pi)$.

3.5.2. Evaluate the integral $\oint_C f(z)\,\mathrm{d}z$, where C is a unit circle centered at the origin, and where $f(z)$ is given below.

(a) $\frac{g(z)}{z-w}$, $g(z)$ entire

(b) $\frac{z}{z^2-w^2}$

(c) ze^{1/z^2}

Solution: We will use characterizations of singularities to address these problems, although an equally acceptable (and frequently simpler) way is to use the Cauchy integral formula. We recall from the solution of problem 3.3.4 that if f has a Laurent series around 0, then the integral of f around a simple closed contour C lying within the region of convergence of the series equals the c_{-1} coefficient of the series. In addition, we assume in what follows that w either is enclosed by C, or lies outside it. (If w lies on C, then the integrals do not have convergent values.)

(a) Since g is entire, then we may expand it in the Taylor series,

$$g(z) = \sum_{j \geq 0} \frac{g^{(j)}(w)}{j!} (z-w)^j.$$

which is uniformly convergent on and in a neighborhood of C. Then using the fact that for an arbitrary $k \in \mathbb{Z}$, $\oint_C (z-w)^k \, \mathrm{d}z = 2\pi i$ if and only if both $k = -1$ and C encloses w (and is zero otherwise), then

$$\oint_C \frac{g(z)}{z-w} \, \mathrm{d}z = \sum_{j \geq 0} \frac{g^{(j)}(w)}{j!} \oint_C (z-w)^{j-1} \, \mathrm{d}z = g(w) \oint_C (z-w)^{-1} \, \mathrm{d}z$$
$$= \begin{cases} 2\pi i g(w), & C \text{ encloses } w \\ 0, & \text{otherwise} \end{cases}$$

(b) If w (and hence also $-w$) lies outside C, then $f(z)$ is only singular at points outside C, and therefore is analytic inside C. Then by Cauchy-Goursat, $\oint_C f(z) \, \mathrm{d}z = 0$. If instead w is enclosed by C, then we use partial fractions along with a Laurent series expansion around 0 using the geometric series, valid for $|z| > |w|$ and thus in particular on C:

$$\frac{z}{z^2 - w^2} = \frac{1/2}{z-w} + \frac{1/2}{z+w} = \left(\frac{1}{2z}\right) \frac{1}{1 - \frac{w}{z}} + \left(\frac{1}{2z}\right) \frac{1}{1 + \frac{w}{z}}$$
$$= \frac{1}{2z} \sum_{j \geq 0} \left(\frac{w}{z}\right)^j + \frac{1}{2z} \sum_{j \geq 0} \left(-\frac{w}{z}\right)^j.$$

Again we have $\oint_C z^k \, \mathrm{d}z = 0$ for $k < -1$, so only the $j = 0$ terms (corresponding to simple poles of f) produce non-zero contribution by integrating over C:

$$\oint_C \frac{z}{z^2 - w^2} \, \mathrm{d}z = \oint_C \frac{1}{2z} \, \mathrm{d}z + \oint_C \frac{1}{2z} \, \mathrm{d}z = 2\pi i.$$

(c) The function e^{1/z^2} has an essential singularity at $z = 0$. Using a Laurent expansion around 0 (valid on C), then:

$$\oint_C z e^{1/z^2} \, \mathrm{d}z = \oint_C \sum_{j \geq 0} \frac{z^{1-2j}}{j!} \, \mathrm{d}z = \sum_{j \geq 0} \frac{1}{j!} \oint_C z^{1-2j} \, \mathrm{d}z = \oint_C z^{-1} \, \mathrm{d}z = 2\pi i.$$